Practice Problems 6 : Differentiability and Rolle's theorem

1. Which of the following functions are differentiable at $x = 0$?
   (a) $f(x) = x^{\frac{1}{3}}$.
   (b) $f(x) = x^2$ for rational x and $f(x) = 0$ for irrational x.
   (c) $f(x) = xsinxcos\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable only at $x = 1$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$.
   (a) If $f(x_0) \neq 0$, show that $|f|$ is also differentiable at x_0.
   (b) If $f(x_0) = 0$, give examples to show that $|f|$ may or may not be differentiable at x_0.

4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$. Define $h(x) = max\{f(x), g(x)\} \ \forall \ x \in \mathbb{R}$. Show that h is differentiable at x_0 if $f(x_0) \neq g(x_0)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = 1$, $f(1) = 1$ and $k \in \mathbb{N}$. Show that
   $$\lim_{n\to\infty} n \left(f(1 + \tfrac{1}{n}) + f(1 + \tfrac{2}{n}) + ... + f(1 + \tfrac{k}{n}) - k \right) = \tfrac{k(k+1)}{2} f'(1).$$

6. Let $f : [0, 1] \to \mathbb{R}$ be differentiable and $f(0) = 0$ and $f(1) = 1$. Show that the equation $f'(x) = 2x$ has a solution on $(0, 1)$.

7. Find the number of real solutions of the following equations.
   (a) $2x - cos^2x + \sqrt{7} = 0$
   (b) $x^{17} - e^{-x} + 5x + cosx = 0$
   (c) $x^{18} + e^{-x} + 5x^2 - 2cosx = 0$.

8. Let $f : [a, b] \to \mathbb{R}$ be such that $f'''(x)$ exists for all $x \in [a, b]$. Suppose that $f(a) = f(b) = f'(a) = f'(b) = 0$. Show that the equation $f'''(x) = 0$ has a solution.

9. Let $a_1, a_2, ..., a_n$ be real numbers such that $a_1 + a_2 + ... + a_n = 0$. Show that the polynomial $q(x) = a_1 + 2a_2x + 3a_3x^2 + ... + na_nx^{n-1}$ has at least one real root.

10. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Suppose that $f'(x)g(x) \neq f(x)g'(x)$ for any $x \in \mathbb{R}$. Show that between any two real solutions of $f(x) = 0$, there is at least one real solution of $g(x) = 0$.

11. Let $f : [0, 1] \to \mathbb{R}$ be differentiable function such that $f(0) = 0$ and $f(x) > 0$ for all $x \in (0, 1]$. Show that there exists $c \in (0, 1)$ such that $\frac{f'(1-c)}{f(1-c)} = \frac{2f'(c)}{f(c)}$.

12. Let $P(x)$ be a polynomial of degree $n, n > 1$ and $P(x_0) = 0$ for some $x_0 \in \mathbb{R}$.
    (a) Show that $P(x) = (x - x_0)Q(x)$ where $Q(x)$ is a polynomial of degree $n - 1$.
    (b) Show that $P'(x_0) = 0$ if and only if $P(x) = (x - x_0)^2R(x)$ where $R(x)$ is a polynomial of degree $n - 2$.
    (c) Show that if all roots of $P(x)$ are real then all roots of $P'(x)$ are also real.

13. **(*)** Let $f : (0, \infty) \to \mathbb{R}$ satisfy $f(xy) = f(x) + f(y)$ for all $x, y \in (0, \infty)$. Suppose that f is differentiable at $x = 1$. Show that f is differentiable at every $x \in (0, \infty)$ and $f'(x) = \frac{1}{x} f'(1)$ for every $x \in (0, \infty)$.

Hints/Solutions

1. (a) Note that $\lim_{x\to 0}\frac{x^{\frac{1}{3}}-0}{x-0}$ does not exist.

   (b) Observe that $\left|\frac{f(x)-0}{x-0}\right| \leq |x| \to 0$ as $x \to 0$. Therefore f is differentiable at $x = 0$.

   (c) Since $\left|\frac{x sinx cos\frac{1}{x}}{x}\right| \leq |sinx| \to 0$, as $x \to 0$, f is differentiable at $x = 0$.

2. Define $f(x) = (x-1)^2$ for rational x and $f(x) = 0$ for irrational x.

3. (a) If $f(x_0) > 0$, then $|f(x)| = f(x)$ in a neighborhood of x_0.

   (b) Consider the examples: (i) $f(x) = x$ (ii) $g(x) = x|x|$.

4. If $f(x_0) > g(x_0)$ then in a neighborhood of x_0, $h(x) = f(x)$.

5. The given limit is $\lim_{n\to\infty}\left(\frac{f(1+\frac{1}{n})-f(1)}{\frac{1}{n}} + 2\frac{f(1+\frac{2}{n})-f(1)}{\frac{2}{n}} + ... + k\frac{f(1+\frac{k}{n})-f(1)}{\frac{k}{n}}\right)$.

6. Apply Rolle's Theorem for $g(x) = f(x) - x^2$ on $[0, 1]$.

7. (a) Let $f(x) = 2x - cos^2x + \sqrt{7}$. Since $f'(x)$ has no real root, by Rolle's theorem $f(x)$ has at most one real root. Now $f(0) > 0$ and $f(-2) < 0$. So by IVP there exists a real solution for $f(x) = 0$. Therefore $f(x) = 0$ has exactly one real solution.

   (b) Let $f(x) = x^{17} - e^{-x} + 5x + cosx$. Observe that $f'(x) > 0 \ \forall \ x \in \mathbb{R}$, $f(2) > 0$ and $f(-2) < 0$. By IVP and Rolle's theorem $f(x) = 0$ has exactly one real solution.

   (c) Let $f(x) = x^{18} + e^{-x} + 5x^2 - 2cosx$. Since $f''(x) > 0 \ \forall \ x \in \mathbb{R}$, $f'(x)$ has at most one real root. Note that $f(0) < 0$, $f(2) > 0$ and $f(-2) > 0$. Therefore by IVP and Rolle's theorem $f(x) = 0$ has exactly two real solutions.

8. By Rolle's theorem there exists $d \in (a, b)$ such that $f'(d) = 0$. Again, by applying Rolle's theorem for f'', there exists $c_1 \in (a, d)$ and $c_2 \in (d, b)$ such that $f''(c_1) = 0$ and $f''(c_2) = 0$. Apply Rolle's Theorem for f'' on $[c_1, c_2]$.

9. Let $p(x) = a_1x + a_2x^2 + ... + a_nx^n$. Then $p(0) = 0$ and $p(1) = 0$. By Rolle's theorem, $p'(x) = q(x)$ has a real root.

10. Let $f(a) = f(b) = 0$ and $a < b$. Since $f'(a)g(a) \neq f(a)g'(a)$, $g(a) \neq 0$. Similarly $g(b) \neq 0$. If $g(x) = 0$ has no real solution then $h(x) = \frac{f(x)}{g(x)}$ is well defined and $h(a) = h(b) = 0$. By Rolle's theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. That is $f'(c)g(c) = f(c)g'(c)$ which is a contradiction.

11. Let $g(x) = (f(x))^2 f(1-x)$. Then $g(0) = g(1) = 0$. Apply Rolle's theorem for g on $[0, 1]$.

12. (a) Use $P(x) = P(x) - P(x_0)$ and $x^k - x_0^k = (x - x_0)(x^{k-1} + x^{k-2}x_0 + ... + xx_0^{k-2} + x_0^{k-1})$.

    (b) Suppose $P(x) = (x-x_0)Q(x)$. Then $P'(x) = Q(x) + (x-x_0)Q'(x)$. If $P'(x_0) = 0$ then $Q(x_0) = 0$. Therefore $Q(x) = (x - x_0)R(x)$ for a polynomial $R(x)$ of degree $n - 2$.

    (c) First observe that if x_0 is a zero of $P(x)$ of order k then it is a zero of $P'(x)$ of order $k - 1$. Use Rolle's theorem that between any two real zeros of $P(x)$ there is one real zero of $P'(x)$.

13. Observe that $f(1) = 0, f(\frac{1}{x}) = -f(x)$ and $f(\frac{x}{y}) = f(x) - f(y)$. Now
    $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
    $= \lim_{h\to 0}\frac{1}{h}f\left(\frac{x+h}{x}\right) = \lim_{k\to 0}\frac{f(1+k)}{kx} = \lim_{k\to 0}\frac{1}{x}\frac{f(1+k)-f(1)}{k} = \frac{1}{x}f'(1)$.