

ROLLE'S THEOREM AND THE MEAN VALUE THEOREM

WILLIAM A. LAMPE

Recall the

Theorem on Local Extrema. *If $f(c)$ is a local extremum, then either f is not differentiable at c or $f'(c) = 0$. That is, at a local max or min f either has no tangent, or f has a horizontal tangent there.*

We will use this to prove

Rolle's Theorem. *Let $a < b$. If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) and $f(a) = f(b)$, then there is a c in (a, b) with $f'(c) = 0$. That is, under these hypotheses, f has a horizontal tangent somewhere between a and b .*

Rolle's Theorem, like the Theorem on Local Extrema, ends with $f'(c) = 0$. The proof of Rolle's Theorem is a matter of examining cases and applying the Theorem on Local Extrema.

Proof. We seek a c in (a, b) with $f'(c) = 0$. That is, we wish to show that f has a horizontal tangent somewhere between a and b .

Since f is continuous on the closed interval $[a, b]$, the Extreme Value Theorem says that f has a maximum value $f(M)$ and a minimum value $f(m)$ on the closed interval $[a, b]$. Either $f(M) = f(m)$ or $f(M) \neq f(m)$.

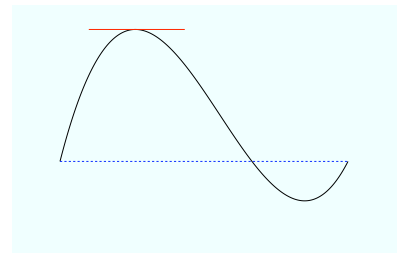
Case 1. We suppose the maximum value $f(M) = f(m)$, the minimum value. So all values of f on $[a, b]$ are equal, and f is constant on $[a, b]$. Then $f'(x) = 0$ for all x in (a, b) . So one may take c to be anything in (a, b) ; for example, $c = \frac{a+b}{2}$ would suffice.

Case 2. Now we suppose $f(M) \neq f(m)$. So at least one of $f(M)$ and $f(m)$ is **not** equal to the value $f(a) = f(b)$.

Case 2.a

We first consider the case where the maximum value $f(M) \neq f(a) = f(b)$. (See the figure to the right.)

So M is neither a nor b . But M is in $[a, b]$ and not at the end points. So M must be in the open interval (a, b) . We have the maximum value $f(M) \geq f(x)$ for all x in the closed interval $[a, b]$ which contains the open interval (a, b) . So we also have $f(M) \geq f(x)$ for every x in the open interval (a, b) . Since M is also in the open interval (a, b) , this means by definition that $f(M)$ is a local maximum.

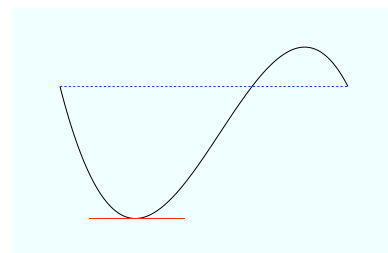


Since M is in the open interval (a, b) , by hypothesis we have that f is differentiable at M . Now by the Theorem on Local Extrema, we have that f has a horizontal tangent at m ; that is, we have that $f'(M) = 0$. So we take $c = M$, and we are done with this case.

Case 2.b

We now consider the case where the minimum value $f(m) \neq f(a) = f(b)$. (This case is very similar to the previous case. Also, see the figure to the right.)

So m is neither a nor b . But m is in $[a, b]$ and not at the endpoints. So m must be in the open interval (a, b) . We have the minimum value $f(m) \leq f(x)$ for all x in the closed interval $[a, b]$ which contains the open interval (a, b) . Thus $f(m) \leq f(x)$ for every x in the open interval (a, b) . Since m is also in the open interval (a, b) , this means by definition that $f(m)$ is a local minimum.



Since m is in the open interval (a, b) , by hypothesis we have that f is differentiable at m . Now by the Theorem on Local Extrema, we have that f has a horizontal tangent at m ; that is, we have that $f'(m) = 0$. So we take $c = m$, and we are done with this case.

Our list of cases covers all possibilities which ends the proof.

Next we give an application of Rolle's Theorem and the Intermediate Value Theorem.

Example. We show that $x^5 + 4x = 1$ has exactly one solution. Let $f(x) = x^5 + 4x$. Since f is a polynomial, f is continuous everywhere. $f'(x) = 5x^4 + 4 \geq 0 + 4 = 4 > 0$ for all x . So $f'(x)$ is never 0. So by Rolle's Theorem, no equation of the form $f(x) = C$ can have 2 or more solutions. In particular $x^5 + 4x = 1$ has at most one solution.

$f(0) = 0^5 + 4 \cdot 0 = 0 < 1 < 5 = 1 + 4 = f(1)$. Since f is continuous everywhere, by the Intermediate Value Theorem, $f(x) = 1$ has a solution in the interval $[0, 1]$.

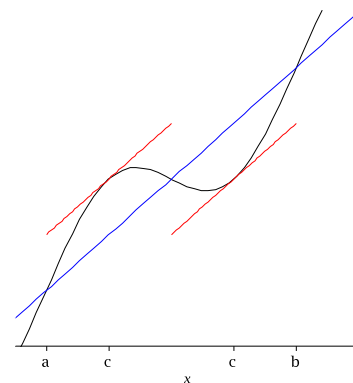
Together these results say $x^5 + 4x = 1$ has exactly one solution, and it lies in $[0, 1]$.

The traditional name of the next theorem is the Mean Value Theorem. A more descriptive name would be Average Slope Theorem.

Mean Value Theorem. *Let $a < b$. If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is a c in (a, b) with*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The Mean Value Theorem says that under appropriate smoothness conditions the slope of the curve at some point between a and b is the same as the slope of the line joining $\langle a, f(a) \rangle$ to $\langle b, f(b) \rangle$. The figure to the right shows two such points, each labeled c .



If f satisfies the hypotheses of the Rolle's Theorem, then the Mean Value theorem also applies and $f(b) - f(a) = 0$. For the c given by the Mean Value Theorem we have $f'(c) = \frac{f(b)-f(a)}{b-a} = 0$. So the Mean Value Theorem says nothing new in this case, but it does add information when $f(a) \neq f(b)$.

The proof of the Mean Value Theorem is accomplished by finding a way to apply Rolle's Theorem. One considers the line joining the points $\langle a, f(a) \rangle$ and $\langle b, f(b) \rangle$. The difference between f and that line is a function that turns out to satisfy the hypotheses of Rolle's Theorem, which then yields the desired result.

Proof. Suppose f satisfies the hypotheses of the Mean Value Theorem. The line joining the points $\langle a, f(a) \rangle$ and $\langle b, f(b) \rangle$ has equation

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

We let g be the difference between f and this line.

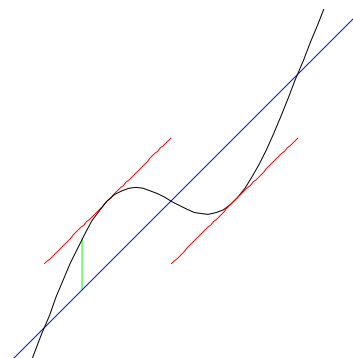
That is,
$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right].$$

$g(x)$ is the height of the vertical green line in the figure to the right.

g is the difference of two continuous functions. So g is continuous on $[a, b]$.

g is the difference of two differentiable functions. So g is differentiable on (a, b) . Moreover, the derivative of g is the difference between the derivative of f and the derivative (slope) of the line. That is,

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$



Both f and the line go through the points $\langle a, f(a) \rangle$ and $\langle b, f(b) \rangle$. So the difference between them is 0 at a and at b . Indeed,

$$g(a) = f(a) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(a - a) \right] = f(a) - [f(a) + 0] = 0, \quad \text{and}$$

$$g(b) = f(b) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(b - a) \right] = f(b) - [f(a) + f(b) - f(a)] = 0.$$

So Rolle's Theorem applies to g . So there is a c in the open interval (a, b) with $g'(c) = 0$. Above we calculated $g'(x)$. Using that we have

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

which is what we needed to prove.

Example. We illustrate The Mean Value Theorem by considering $f(x) = x^3$ on the interval $[1, 3]$.

f is a polynomial and so continuous everywhere. For any x we see that $f'(x) = 3x^2$. So f is continuous on $[1, 3]$ and differentiable on $(1, 3)$. So the Mean Value theorem applies to f and $[1, 3]$.

$$\frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(1)}{3 - 1} = \frac{27 - 1}{2} = 13.$$

$f'(c) = 3c^2$. So we seek a c in $[1, 3]$ with $3c^2 = 13$.

$$3c^2 = 13 \text{ iff } c^2 = \frac{13}{3} \text{ iff } c = \pm\sqrt{\frac{13}{3}}.$$

$-\sqrt{\frac{13}{3}}$ is not in the interval $(1, 3)$, but $\sqrt{\frac{13}{3}}$ is a little bigger than $\sqrt{\frac{12}{3}} = \sqrt{4} = 2$. So $\sqrt{\frac{13}{3}}$ is in the interval $(1, 3)$.

So $c = \sqrt{\frac{13}{3}}$ is in the interval $(1, 3)$, and

$$f'(c) = f' \left(\sqrt{\frac{13}{3}} \right) = 13 = \frac{f(3) - f(1)}{3 - 1} = \frac{f(b) - f(a)}{b - a}.$$