CHAPTER-2

Solutions of Linear Partial Differential Equations of Order One

2.1 Introduction

Partial differential equations of order one arise in many practical problems in science and engineering, when the number of independent variables in the problem under discussion is two or more. The most general form of a partial differential equation of order one in two independent variables x and y and a dependent variable z is $f(x, y, z, p, q) = 0$, where $p \equiv \frac{\partial z}{\partial x}$ and $q \equiv \frac{\partial z}{\partial y}$. In this chapter, we shall consider only linear partial differential equations of order one.

2.2 Linear Partial Differential Equation of Order One

A partial differential equation $f(x, y, z, p, q) = 0$ of order one is said to be linear, if it is of first degree in p and q. There is no restriction on the degree of the dependent variable z. For example, the equations

$$xp + yq = xy \quad \text{and} \quad x^2p + y^2q = z^2 \qquad \ldots(1)$$

are linear partial differential equations of order one.

The general form of a linear partial differential equation of order one is

$$Pp + Qq = R \qquad \ldots(2)$$

where P, Q and R are functions of x, y and z.

If P = 0 or Q = 0 in (2), then the equation can be solved easily. For example, the equation $\frac{\partial z}{\partial y} = 3x + 4y$ has its solution $z = 3xy + 2y^2 + f(x)$, where f is an arbitrary function of *x*. Similarly, the equation $\frac{\partial z}{\partial x} = 2x - 3y$ has its solution $z = x^2 - 3xy + g(y)$, where g is an arbitrary function of *y*.

2.3 Classification of Partial Differential Equations of Order One

The partial differential equations of order one may be classified as under:

2.3.1 Quasi-linear Partial Differential Equation

A partial differential equation of order one of the form

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z) \qquad \ldots(1)$$

is called a **quasi-linear partial differential equation of order one**, if the degree of partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ appearing in the equation is one and the coefficients P, Q and R depend upon *x*, *y* and *z*, e.g. the partial differential equations $z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ and $y\frac{\partial z}{\partial x} + xz\frac{\partial z}{\partial y} = xyz$ are quasi-linear partial differential equations of order one.

2.3.2 Almost-linear Partial Differential Equation

A partial differential equation of order one of the form

$$P(x,y)\frac{\partial z}{\partial x} + Q(x,y)\frac{\partial z}{\partial y} = R(x,y,z) \qquad \ldots(2)$$

is called an **almost linear partial differential equation of order one**, if the coefficients P and Q are functions of the independent variables *x* and *y* only and R is a function of *x*, *y* and *z*; e.g. the partial differential equations $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z^2$ and $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=z^3$ are almost linear partial differential equations of order one.

2.3.3 Linear Partial Differential Equation

A partial differential equation of order one of the form

$$a(x,y)\frac{\partial z}{\partial x}+b(x,y)\frac{\partial z}{\partial y}+c(x,y)z=d(x,y) \qquad \ldots(3)$$

is called a **linear partial differential equation of order one**, if the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and the dependent variable z appear in linear form in the equation while the coefficients *a*,*b*,*c* and *d* depend only on the independent variables *x* and *y*, e.g. the partial differential equations $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ and $y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}+z=xy$ are linear partial differential equations of order one.

2.3.4 Non-linear Partial Differential Equation

A partial differential equation of order one which does not fit into any of the above categories is called **non-linear partial differential equation of order one,** e.g. the partial differential equations $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=1$ and $x^2\left(\frac{\partial z}{\partial x}\right)^2+y^2\left(\frac{\partial z}{\partial y}\right)^2=z^2$ are non-linear partial differential equations of order one.

2.4 Origin of Linear Partial Differential Equations of Order One

Before discussing the solution of the partial differential equations of order one, we shall examine the interesting question of how they arise. For the purpose, let us consider

$$x^2+y^2+(z-c)^2=k^2 \qquad \ldots(1)$$

where *c* and *k* are arbitrary constants. The equation (1) represents the set of all spheres whose centers lie along the z-axis.

Differentiating (1) partially w.r.t. *x*, we get

$$x + p(z - c) = 0 \qquad \ldots(2)$$

Again, differentiating (1) partially w.r.t. *y*, we get

$$y + q(z - c) = 0 \qquad \ldots(3)$$

Eliminating the arbitrary constant *c* from (2) and (3), we get

$$yp - xq = 0 \qquad \ldots(4)$$

which is a partial differential equation of order one.

Thus, we see that the set of all spheres with centers on the z-axis is characterized by the partial differential equation (4). In some sense, the function z defined by the equation (1) is called a solution of the partial differential equation (4).

In chapter 1, we have already seen the origin of partial differential equations of order one.

2.5 Lagrange's Partial Differential Equation of Order One

The quasi-linear partial differential equation of order one of the form

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z) \qquad \ldots(1)$$

i.e.

$$Pp + Qq = R \qquad \ldots(2)$$

where P, Q and R are functions of *x*, *y* and z is known as **Lagrange's partial differential equation**. e.g. For example, $xyp + yzq = xy$ and $y^2p - xyq = x(z - 2y)$ are Lagrange's partial differential equations. For getting the solution of (1) or (2), we wish to find a relation between *x*, *y* and *z* involving an arbitrary function. The first systematic theory of equations of the above type characterized by (1) or (2) is given by Lagrange. For this reason, the partial differential equation (1) or (2) is frequently referred to as **Lagrange's equation**. It should be noted that in this connection the term linear means that p and q appear in the first degree only but P, Q and R may be any functions of *x*, *y* and z.

2.6 Solutions of Linear Partial Differential Equations of Order One

We have already observed that the relation of the form

$$\phi(x, y, z, a, b) = 0 \quad \ldots(1)$$

gives rise to PDE of order one of the from

$$f(x, y, z, p, q) = 0 \quad \ldots(2)$$

Thus, any relation of the form (1) containing two arbitrary constants *a* and *b* is a solution of the PDE (2).

Now, let us consider the following Lagrange's partial differential equation of order one

$$P(x, y, z)\frac{\partial z}{\partial x} + Q(x, y, z)\frac{\partial z}{\partial y} = R(x, y, z) \quad \ldots(3)$$

i.e.,

$$Pp + Qq = R \quad \ldots(4)$$

where *x* and *y* are independent variables. The solution of equation (3) or (4) is a surface S lying in the (*x*, *y*, *z*) –space, and is called as an **integral surface**. If we are given that $z = f(x, y)$ is an integral surface of the PDE (3) or (4), then the normal to this surface will have direction cosines proportional to $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ or $(p, q, -1)$. Therefore, the direction of the normal is given by $\vec{n}= \{p, q, -1\}$. From the PDE (4), we observe that the normal $\vec{n}$ is perpendicular to the direction defined by the vector $\vec{t} = \{P, Q, R\}$ as shown in the Figure 2.1.

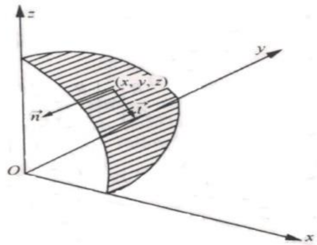

Figure 2.1: Integral Surface z = f(x, y)

Therefore, any integral surface must be tangential to a vector with components $\{P, Q, R\}$ and hence, will never leave the integral surface. Also, the total differential dz is given by

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \qquad \ldots(5)$$

From equations (4) and (5), we find that

$$\{P, Q, R\} = \{dx, dy, dz\} \qquad \ldots(6)$$

2.7 Method of Solution of Lagrange's Partial Differential Equation

We have seen the Lagrange's partial differential equation of the form $Pp + Qq = R$, where P, Q and R are functions of x, y and z.

The method of solution of Lagrange's partial differential equation is contained in following theorem:

Theorem: The general solution of the Lagrange's partial differential equation

$$Pp + Qq = R \qquad \ldots(1)$$

is

$$\phi(u, v) = 0 \qquad \ldots(2)$$

where

$$u(x, y, z) = c_1 \text{ and } v(x, y, z) = c_2 \qquad \ldots(3)$$

are two independent solutions of the following system of auxiliary equations:

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \ldots(4)$$

where ϕ is an arbitrary function, c_1 and c_2 are arbitrary constants and at least one of u and v must contain z.

The set of equations given by (4) are called **Lagrange's auxiliary equations or Lagrange's subsidiary equations.** The curves given by $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ are called the **characteristic curves**.

Proof: Given Lagrange's partial differential equation is

$$Pp+Qq=R \qquad \ldots(1)$$

Let $$\phi(u,v)=0 \qquad \ldots(2)$$

be the solution of the given Lagrange's equation (1).

Differentiating (2) partially w.r.t. x, we get

$$\left(\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right)+\left(\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)=0$$

or $$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0 \qquad \ldots(5)$$

Similarly, differentiating (2) partially w.r.t. y, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0 \qquad \ldots(6)$$

Eliminating ϕ i.e., $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from (5) and (6), we get

$$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)-\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0$$

or $\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q+\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=0$

$$\therefore \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \ldots(7)$$

which can also be put in the form

$$\frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \qquad \ldots(8)$$

or

$$Pp+Qq=R$$

where

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}=\frac{\partial(u,v)}{\partial(y,z)} \qquad \ldots(9)$$

$$Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial(u,v)}{\partial(z,x)} \qquad \ldots(10)$$

and

$$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial(u,v)}{\partial(x,y)} \qquad \ldots(11)$$

Thus, the equation $Pp+Qq=R$ is a partial differential equation of order one and degree one for which $\phi(u,v)=0$ is a solution.

Now, taking the differentials of two independent solutions $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$, we get

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial y}{\partial z}dz=0 \qquad \ldots(12)$$

and

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0 \qquad \ldots(13)$$

Since u and v are independent functions, therefore, solving equations (12) and (13) for the ratios $dx:dy:dz$, we get

$$\frac{dx}{\frac{\partial u\partial v}{\partial y\partial z}-\frac{\partial u\partial v}{\partial z\partial y}}=\frac{dy}{\frac{\partial u\partial v}{\partial z\partial x}-\frac{\partial u\partial v}{\partial x\partial z}}=\frac{dz}{\frac{\partial u\partial v}{\partial x\partial y}-\frac{\partial u\partial v}{\partial y\partial x}} \qquad \ldots(14)$$

Now, comparing equation (14) with (4), we obtain

$$\frac{\frac{\partial u \partial v}{\partial y \partial z}-\frac{\partial u \partial v}{\partial z \partial y}}{P}=\frac{\frac{\partial u \partial v}{\partial z \partial x}-\frac{\partial u \partial v}{\partial x \partial z}}{Q}=\frac{\frac{\partial u \partial v}{\partial x \partial y}-\frac{\partial u \partial v}{\partial y \partial x}}{R}=k, \text{ say} \qquad \ldots(15)$$

$$\therefore \quad \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}=kP,\ \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=kQ,\ \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=kR$$

Substituting these values in equation (7), we get

$$k(Pp+Qq)=kR \quad \text{or} \quad Pp+Qq=R$$

which is the given partial differential equation (1).

Therefore, if $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ are two independent solutions of the system of differential equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, then $\phi(u,v)=0$ is a solution of $Pp+Qq=R$, where ϕ is an arbitrary function.

2.8 General Methods of Solution of Lagrange's Equation

Let us consider the Lagrange's partial differential equation

$$Pp+Qq=R \qquad \ldots(1)$$

The Lagrange's auxiliary equations for (1) are:

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \qquad \ldots(2)$$

which are generally solved by the following two methods:

2.8.1 Method of Grouping

In this method, we take any set of two fractions (ratios) of (2), equate them and cancel the common factor, if any in the denominators. Then, we integrate the resulting differential equation to get a solution of the form $u(x, y, z) = c_1$. Similarly, we take another set of two fractions of (2), equate them and repeat the above procedure to get another solution of the form $v(x, y, z) = c_2$. These two solutions will constitute the general solution in one of the forms ϕ $(u, v) = 0$ or $u = \phi(v)$ or $v = \phi(u)$, where ϕ is an arbitrary function.

2.8.2 Method of Multipliers

In this method, we choose any three multipliers l, m, n which may be constants or functions of *x*, *y* and *z* in such a way that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} \qquad \ldots(3)$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$, then the value of numerator $l\,dx + m\,dy + n\,dx$ in the last fraction of (3) is also zero i.e. $l\,dx + m\,dy + n\,dx = 0$ which can be integrated to have $u(x, y, z) = c_1$. This process may be repeated to have another integral $v(x, y, z) = c_2$. Sometimes the numerator $l\,dx + m\,dy + n\,dz$ is an exact differential of the denominator, then on integration, we get a solution of the form $u\,(x, y, z) = c_1$. This process is repeated to have another solution $v(x,y,z) = c_2$. Finally, the solutions $u(x, y, z) = c_1$and $v(x, y, z) = c_2$ will constitute the general solution in one of the forms $\phi(u, v) = 0$ or $u = \phi(v)$ or $v = \phi(u)$

The multipliers l, m and n are called **Legrange's multipliers** or **Lagrangian multipliers.**

2.8.3 Working Rules for Solving $Pp + Qq = R$ by Lagrange's Method

The following steps are required for solving the given partial differential equation of order one by Lagrange's method:

Step 1. Put the given partial differential equation in the form

$$Pp + Qq = R \qquad \ldots(1)$$

Step 2. Write down Lagrange's auxiliary equations for (1), namely

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots(2)$$

Step 3. Solve (2) either by using the method of grouping or by the method of multipliers.

Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ ...(3)

be the two independent solutions of (2).

Step 4. The general solution (or integral) of (1) is then written in one of the following three equivalent forms:

$$\phi(u, v) = 0, u = \phi(v) \text{ or } v = \phi(u) \quad \ldots(4)$$

where ϕ is an arbitrary function.

2.9 Certain Rules for Solving Lagrange's Auxiliary Equations

Here, we shall discuss four rules for getting two independent solutions of the Lagrange's auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. Accordingly, we have four types of problems based on Lagrange's partial differential equation $Pp + Qq = R$.

2.9.1 Rule I for Solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Let the Lagrange's auxiliary equations for the partial differential equation

$$Pp + Qq = R \quad \ldots(1)$$

be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \ldots(2)$$

If one of the variables is either absent or cancels out from any two fractions of Lagrange's auxiliary equation (2), then in this case, an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of Lagrange's auxiliary equations (2).

The following examples will make the concept more clear:

SOLVED EXAMPLES

Example 1. Solve the partial differential equation $2p + 3q = 1$ by Lagrange's method.

Solution : The given partial differential equation can be written as

$$Pp + Qq = R \quad \ldots(1)$$

where $P = 2,\ Q = 3$ and $R = 1$

The Lagrange's auxiliary equations for (1) are given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{or} \quad \frac{dx}{2} = \frac{dy}{3} = \frac{dz}{1} \quad \ldots(2)$$

Taking the first two fractions of (2), we have

$$\frac{dx}{2} = \frac{dy}{3} \quad \text{or} \quad 3dx - 2dy = 0$$

which on integration gives $3x - 2y = c_1$ ...(3)

$\therefore u(x, y, z) \equiv 3x - 2y = c_1$ is one solution of the given partial differential equation.

Similarly, taking the last two fractions of (2), we have

$$\frac{dy}{3} = \frac{dz}{1} \quad \text{or} \quad dy - 3dz = 0$$

which on integration gives $y - 3z = c_2$ ...(4)

$\therefore v(x, y, z) \equiv y - 3z = c_2$ is another solution of the given partial differential equation.

Hence, the desired general solution is given by

$$\phi(3x - 2y, y - 3z) = 0 \quad \ldots(5)$$

where ϕ is an arbitrary function.

Example 2. Find the general solution of $zp + x = 0$.

Solution : The given partial differential equation can be written as

$$Pp + Qq = R \qquad \ldots(1)$$

where $P = z$, $Q = 0$ and $R = -x$.

The Lagrange's auxiliary equations for (1) are:

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x} \qquad \ldots(2)$$

Taking the first and the last fractions of (2), we have

$$\frac{dx}{z} = \frac{dz}{-x} \quad \text{or} \quad x\,dx + z\,dz = 0$$

which on integration gives $\frac{x^2}{2} + \frac{z^2}{2} = k$ or $x^2 + z^2 = c_1$ ...(3)

$\therefore$ $u(x, y, z) \equiv x^2 + z^2 = c_1$ is one solution of the given partial differential equation.

Also, the second fraction of (2) implies that $dy = 0$

which on integration gives $y = c_2$ ...(4)

$\therefore$ $v(x, y, z) \equiv y = c_2$ is another solution of the given partial differential equation.

Hence, the desired general solution is given by

$$\phi\,(x^2 + z^2, y) = 0 \qquad \ldots(5)$$

where ϕ is an arbitrary function.

Example 3. Solve $p \tan x + q \tan y = \tan z$.

Solution. Given that $(\tan x)p + (\tan y)q = \tan z$ ...(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \qquad \ldots(2)$$

Taking the first two fractions of (2), we get

$$\cot x\, dx - \cot y\, dy = 0$$

which on integration gives $\log \sin x - \log \sin y = \log c_1$

or $\log\left(\frac{\sin x}{\sin y}\right) = \log c_1$ or $\frac{\sin x}{\sin y} = c_1$ …(3)

Taking the last two fractions of (2), we get

$$\cot y\, dy - \cot z\, dz = 0$$

which on integration gives $\log \sin y - \log \sin z = \log c_2$

or $\log\left(\frac{\sin y}{\sin z}\right) = \log c_2$ or $\frac{\sin y}{\sin z} = c_2$ …(4)

From (3) and (4), the required general solution is given by

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0 \qquad \ldots(5)$$

where ϕ is an arbitrary function.

Example 4. Solve $y^2p - xyq = x(z - 2y)$.

Solution. Given that $y^2p - xyq = x(z - 2y)$ …(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y^2} = \frac{dx}{-xy} = \frac{dz}{x(z-2y)} \qquad \ldots(2)$$

Taking the first two fractions of (2), we get $\frac{dx}{y} = \frac{dy}{-x}$

or $2xdx + 2ydy = 0$ so that $x^2 + y^2 = c_1$ …(3)

Taking the last two fractions of (2), we get $\frac{dy}{-y} = \frac{dz}{z-2y}$

or $\frac{dz}{dy} = -\frac{z-2y}{y}$ or $\frac{dz}{dy} + \left(\frac{1}{y}\right) z = 2$ …(4)

which is a linear differential equation in z.

I.F. of (4) is given by I.F.$= e^{\int (1/y) dy} = e^{\log y} = y$

∴ The solution of equation (4) is given by

$$zy = \int 2y\, dy + c_2 \quad \text{or} \quad zy - y^2 = c_2 \qquad \text{…(5)}$$

∴ From (3) and (5), the desired solution is given by

$$\phi(x^2 + y^2, zy - y^2) = 0 \qquad \text{…(6)}$$

where ϕ is an arbitrary function.

Example 5. Solve $(x^2 + 2y^2)p - xyq = xz.$

Solution. Given that $(x^2 + 2y^2)p - xyq = xz$ …(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x^2+2y^2} = \frac{dy}{-xy} = \frac{dz}{xz} \qquad \text{…(2)}$$

Taking the last two fractions of (2), we get

$$\frac{dy}{-y} = \frac{dz}{z} \quad \text{or} \quad \frac{dy}{y} + \frac{dz}{z} = 0$$

which on integration gives $\log y + \log z = \log c_1$

or $\log yz = \log c_1$ or $yz = c_1$ …(3)

Again, taking the first two fractions of (2), we have

$$\frac{dx}{dy} = \frac{x^2+2y^2}{-xy} \quad \text{or} \quad 2x\frac{dx}{dy} + \left(\frac{2}{y^2}\right) x^2 = -4y \qquad \text{…(4)}$$

Putting $x^2 = v$ so that $2x\frac{dx}{dy} = \frac{dv}{dy}$ in (4), we get

$$\frac{dv}{dx} + \left(\frac{2}{y}\right)v = -4y \qquad \ldots(5)$$

which is a linear differential equation in *v*.

Its integrating factor $= e^{\int (2/y)dy} = e^{2\log y} = y^2$

Therefore, the solution of equation (5) is given by

$$y^2 v = \int\{(-4y)xy^2\}\,dy + c_2 \quad \text{or} \quad y^2x^2 + y^4 = c^2 \qquad \ldots(6)$$

Hence, from (3) and (6), the general solution is given by

$$\phi(yz, y^2x^2 + y^4) = 0 \qquad \ldots(7)$$

where ϕ is an arbitrary function.

EXERCISE 2(A)

Solve the following partial differential equations:

1. $p + q = 1$ **2.** $xp + yq = z$ **3.** $zp = x$

4. $x^2p + y^2p = z^2$ **5.** $x^2p + y^2q + z^2 = 0$

6. $p + q = \sin x$ **7.** $yzp + 2xq = xy$ **8.** $yzp + zxq = xy$

ANSWERS

1. $\phi(x - y, x - z) = 0$ **2.** $\phi\left(\frac{x}{z}, \frac{y}{z}\right) = 0$

3. $\phi(y, x^2 - z^2) = 0$ **4.** $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$

5. $\phi\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \frac{1}{z}\right) = 0$ **6.** $\phi(x - y, z + \cos x) = 0$

7. $\phi(x^2 - z^2, y^2 - 4z) = 0$ **8.** $\phi(x^2 - y^2, x^2 - z^2) = 0$

2.9.2 Rule II for Solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Let the Lagrange's auxiliary equations for the partial differential equation $$Pp + Qq = R \quad \ldots(1)$$

be $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \ldots(2)$$

Suppose that one integral of (2) is known by using rule 1 explained in previous article and suppose also that another integral cannot be obtained by using the rule I of previous article. Then, one (the first) integral known to us is used to find another (the second) integral as shown in the following solved examples. Note that in the second integral, the constant of integration of the first integral should be removed later on.

SOLVED EXAMPLES

Example 1. Find the general solution of p + 3q = 5z + tan (y – 3x).

Solution : Given that $$p + 3q = 5z + \tan(y - 3x) \quad \ldots(1)$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)} \quad \ldots(2)$$

Taking the first two functions of (2), we have

$$\frac{dx}{1} = \frac{dy}{3} \quad \text{or} \quad dy - 3dx = 0 \quad \ldots(3)$$

which on integration gives $$y - 3x = c_1 \quad \ldots(4)$$

∴ $y - 3x = c_1$ is one solution of the given PDE, where c_1 is an arbitrary constant.

Again, taking the first and the last fractions of (2), we have

$$\frac{dx}{1} = \frac{dz}{5z+\tan(y-3x)} \quad \text{or} \quad dx = \frac{dz}{5z+\tan(y-3x)}$$

Putting $y - 3x = c_1$ from (4) in it, we obtain $dx = \frac{dz}{5z+\tan c_1}$ ...(5)

which on integration gives $x = \frac{1}{5}\log(5z + \tan c_1) + c_2$ ...(6)

Removing the constant c_1 from this by using (4), we get

$$5x - \log[5z + \tan(y - 3x)] = c_2 \quad \text{...(7)}$$

$\therefore$ $5x - \log[5z + \tan(y - 3x)] = c_2$ is another solution of the given PDE, where c_2 is an arbitrary constant.

Hence, the required general solution is given by

$$\phi[y - 3x, 5x - \log \{5z + \tan(y - 3x)\}] = 0 \quad \text{...(8)}$$

where ϕ is an arbitrary function.

Example 2. Solve $xz(z^2 + xy)p - yz(z^2 + xy)q = x^4$.

Solution. Given that $xz(z^2 + xy)p - yz(z^2 + xy)q = x^4$ ...(1)

The Lagrange's auxiliary equation for (1) are

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4} \quad \text{...(2)}$$

Cancelling $z(z^2 + xy)$ from first two fractions of (2), we get

$$\frac{dx}{x} = \frac{dy}{-y} \quad \text{or} \quad \frac{dx}{x} + \frac{dy}{y} = 0 \quad \text{...(3)}$$

Integrating (3), we get $\log x + \log y = \log c_1$ or $xy = c_1$ ...(4)

Taking the first and the last fractions of (2), we have

$$\frac{dz}{xz(z^2+xy)} = \frac{dz}{x^4}$$

Putting $xy = c_1$ from (4) in it, we get $\frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4}$

or $x^3dx = z(z^2 + c_1)dz$ or $x^3dx - (z^3 + c_1z)dz = 0$ …(5)

Integrating (5), we get $\frac{x^4}{4} - \frac{z^4}{4} - \frac{c_1z^2}{2} = \frac{c_2}{4}$

or $$x^4 - z^4 - 2c_1z^2 = c_2 \quad \ldots(6)$$

Removing the constant c_1 from this by using (4), we get

$$x^4 - z^4 - 2xyz^2 = c_2 \quad \ldots(7)$$

From (4) and (7), the required integral is given by

$$\phi(xy, x^4 - z^4 - 2xyz^2) = 0 \quad \ldots(8)$$

where ϕ is an arbitrary function.

Example 3. Solve $xyp + y^2q = zxy - 2x^2$.

Solution. Given that $xyp + y^2q = zxy - 2x^2$ …(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy-2x^2} \quad \ldots(2)$$

Taking the first two fractions of (2), we have

$$\frac{dx}{xy} = \frac{dy}{y^2} \quad \text{or} \quad \frac{1}{x}dx - \frac{1}{y}dy = 0 \quad \ldots(3)$$

Integrating (3), we get $\log x - \log y = \log c_1$

or $$x/y = c_1 \quad \text{or} \quad x = c_1y \quad \ldots(4)$$

Again, taking the last two fractions of (2), we get

$$\frac{dy}{y^2} = \frac{dz}{zxy-2x^2}$$

Putting $x = c_1 y$ from (4) in it, we get $\frac{dy}{y^2} = \frac{dz}{c_1 z y^2 - 2c_1^2 y^2}$

or $$dy = \frac{dz}{c_1(z-2c_1)} \quad \text{or} \quad c_1 dy - \frac{dz}{z-2c_1} = 0 \qquad \ldots(5)$$

Integrating (5), we get $c_1 y - \log(z - 2c_1) = c_2$ ...(6)

Removing constant c_1 from this by using (4), we get

$$x - \log[z - 2(x/y)] = c_2 \qquad \ldots(7)$$

From (4) and (7), the required general solution is given by

$$\phi[(x/y), x - \log\{z - 2(x^2/y^2)\}] = 0 \qquad \ldots(8)$$

where ϕ is an arbitrary function.

Example 4. Solve $xzp + yzq = xy.$

Solution. Given that $xzp + yzq = xy$ ...(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy} \qquad \ldots(2)$$

Taking the first two fractions of (2), we get

$$\frac{dx}{x} - \frac{dy}{y} = 0 \qquad \ldots(3)$$

Integrating (3), we get $\log x - \log y = \log c_1$

or $$x/y = c_1 \quad \text{or} \quad x = c_1 y \qquad \ldots(4)$$

Taking the last two fractions of (2), we get $\frac{dy}{yz} = \frac{dz}{xy}$

Using $x = c_1 y$ from (4) in it, we get $\frac{dy}{yz} = \frac{dz}{c_1 y^2}$

or $c_1 ydy = zdz$ or $2c_1 ydy - 2z\,dz = 0$ …(5)

Integrating (5), we get $c_1 y^2 - z^2 = c_2$ …(6)

Removing constant c_1 from this by using (4), we get

$$xy - z = c_2 \quad \ldots(7)$$

From (4) and (7), the required solution is given by

$$\phi(x/y, xy - z^2) = 0 \quad \ldots(8)$$

where ϕ is an arbitrary function.

Example 5. Solve $py + qx = xyz^2(x^2 - y^2)$.

Solution. Given that $py + qx = xyz^2(x^2 - y^2)$ …(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)} \quad \ldots(2)$$

Taking the first two fractions of (2), we get $\frac{dx}{y} = \frac{dy}{x}$

or $x\,dy - y\,dy = 0$ or $2xdx - 2ydy = 0$ …(3)

Integrating it, we get $x^2 - y^2 = c_1$ …(4)

Taking the last two fractions of (2), we get $\frac{dy}{x} = \frac{dz}{xyz^2(x^2-y^2)}$

Using (4) in it, we get $\frac{dy}{x} = \frac{dz}{xyz^2 c_1}$

or $$2c_1 ydy - 2z^{-2}dz = 0 \quad \ldots(5)$$

Integrating (5), we get $c_1 y^2 + \left(\frac{2}{z}\right) = c_2$ …(6)

Removing constant c_1 from this by using (4), we get

$$y^2(x^2 - y^2) + (2/z) = c_2 \qquad \ldots(7)$$

From (4) and (7), the required solution is given by

$$\phi[(x^2 - y^2), y^2(x^2 - y^2) + (2/z)] = 0 \qquad \ldots(8)$$

where ϕ is an arbitrarty function.

Example 6. Solve $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$.

Solution. Re-writing the given partial differential equation, we have

$$x(z - 2y^2)p + y(z - y^2 - 2x^3)q = z(z - y^2 - 2x^3) \quad \ldots(1)$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \qquad \ldots(2)$$

Taking the last two fractions of (2), we get $\frac{dz}{z} = \frac{dy}{y}$

Integrating it, we get $\qquad \log z = \log y + \log c_1$

or $\qquad z/y = c_1 \qquad$ or $\qquad z = c_1 y \qquad \ldots(3)$

where c_1 is an arbitrary constant.

Again, taking the first two fractions of (2), we have

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)}$$

Using (3) in it, we get $\frac{dx}{x(c_1y-2y^2)} = \frac{dy}{y(c_1y-y^2-2x^3)}$

or $\qquad (c_1y - y^2 - 2x^3)dx + x(2y - c_1)dy = 0 \qquad \ldots(4)$

Comparing (4) with $Mdx + Ndy = 0$, we have

$$M = c_1y - y^2 - 2x^3 \text{ and } N = x(2y - c_1)$$

$\therefore \quad \dfrac{\partial M}{\partial y} = c_1 - 2y \qquad$ and $\dfrac{\partial N}{\partial x} = 2y - c_1$

Now $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) = \dfrac{1}{x(2y-c_1)}[(c_1 - 2y) - (2y - c_1)]$

$$= \frac{-2(2y-c_1)}{x(2y-c_1)} = -\frac{2}{x}$$

which is a function of x alone.

Hence, the integrating factor (I.F.) of (4) is given by

$$\text{I.F.} = e^{\int(-2/x)dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$$

Multiplying (4) by x^{-2}, we get the following equation:

$$(c_1 y x^{-2} - y^2 x^{-2} - 2x)dx + x^{-1}(2y - c_1)dy = 0$$

By usual rule, its solution is given by

$$\int\{(c_1 y - y^2)x^{-2} - 2x\}\,dx + \int x^{-1}\,(2y - c_1)dy = c_2$$

(Treating y as constant) (Integrating terms free from x)

or $(c_1 y - y^2)(-1/x) - x^2 = c_2$ or $(y^2 - c_1 y)/x - x^2 = c_2$

Removing constant c_1 from this by using (3), we get

$$(y^2 - z - x^3)/x = c_2, \text{ since } c_1 y = z \qquad \ldots(5)$$

where c_2 is an arbitrary constant.

From (3) and (5), the required solution is given by

$$\phi[(z/y), (y^2 - z - x^3)/x] = 0 \qquad \ldots(6)$$

where ϕ is an arbitrary function.

EXERCISE 2(B)

Solve the following partial differential equations:

1. $p - 2q = 3x^2 \sin(y + 2x)$ **2.** $p - q = z/(x + y)$

3. $xy^2p - y^3q + axz = 0$ **4.**$(x^2 - y^2 - z^2)p + 2xyq = 2x$

5. $z(p - q) = z^2 + (x + y)^2$ **6.** $p + 3q = z + \cot(y - 3x)$

7. $xyp + y^2q = xyz - 2x^2$ **8.** $zp - zq = x + y$

ANSWERS

1. $\phi[y + 2x, x^2 \sin(y + 2x) - z] = 0$

2. $\phi[x + y, x - (x + y) \log z] = 0$

3. $\phi[xy, \log z\,(ax/3y^2)] = 0$

4. $\phi[y/z, (x^2 + y^2 + z^2)/z] = 0$

5. $\phi[x + y, e^{2y}\{z^2 + (x + y)^2\}] = 0$

6. $\phi[y - 3x, x - \log|z + \cot(y - 3x)|] = 0$

7. $\phi[x/y, x - \log|z - (2x/y)|] = 0$

8. $\phi[x + y, 2x(x + y) - z^2] = 0$

2.9.3 Rule III for Solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Let the Lagrange's auxiliary equations for the partial differential equation $Pp + Qq = R$ …(1)

be $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{…(2)}$$

Again, if P_1, Q_1 and R_1 be functions of x, y and z, then by a well-known principle of algebra, each fraction in (2) will be equal to

$$(P_1 dx + Q_1 dy + R_1 dz)/(P_1 P + Q_1 Q + R_1 R) \qquad \ldots(3)$$

If $P_1 P + Q_1 Q + R_1 R = 0$, then the numerator of (3) is also zero. This gives $P_1 dx + Q_1 dy + R_1 dz = 0$ which can be integrated to give $u(x, y, z) = c_1$. This method may be repeated to get another integral $v(x, y, z) = c_2$. Here, P_1, Q_1, and R_1 are called as **Lagrange's multipliers**. As a special case, these can be constants also. Sometimes, only one integral is possible by the use of Lagrange's multipliers. In such cases, the second integral should be obtained either by using rule I or rule II of the previous articles as the case may be.

SOLVED EXAMPLES

Example 1. Solve $\{(b-c)/a\}yzp + \{(c-a)/b\}zxp = \{(a-b)/c\}xy$

Solution. Given partial differential equation is

$$\{(b-c)/a\}yzp + \{(c-a)/b\}zxp = \{(a-b)/c\}xy \qquad \ldots(1)$$

The Lagrange's auxiliary equations of (1) are

$$\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)zx} = \frac{c\,dz}{(a-b)xy} \qquad \ldots(2)$$

Choosing x, y and z as multipliers, each fraction for (2) is

$$= \frac{ax\,dx + by\,dy + cz\,dz}{xyz[(b-c)+(c-a)+(a-b)]} = \frac{ax\,dx + by\,dy + cz\,dz}{0}$$

$\therefore \quad ax\,dx + by\,dy + cz\,dz = 0$ or $\quad 2ax\,dx + 2bydy + 2czdz = 0$

Integrating it, we get $\qquad ax^2 + by^2 + cz^2 = c_1 \qquad \ldots(3)$

where c_1 is an arbitrary constant.

Choosing ax, by and cz as multipliers, each fraction of (2) is

$$= \frac{a^2xdx+b^2ydy+c^2zdz}{xyz[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2xdx+b^2ydy+c^2zdz}{0}$$

$\therefore\ a^2xdx + b^2ydy + c^2zdz = 0$ or $2a^2xdx + 2b^2ydy + 2c^2zdz = 0$

Integrating it, we get $\quad a^2x^2 + b^2y^2 + c^2z^2 = c_2 \qquad \ldots(4)$

where c_2 is an arbitrary constant.

From (3) and (4), the required general solution is given by

$$\phi(ax^2 + by^2 + cz^2, a^2x^2 + b^2y^2 + c^2z^2) = 0 \qquad \ldots(5)$$

where ϕ is an arbitrary function.

Example 2. Solve $\quad z(x+y)p + z(x-y)q = x^2 + y^2.$

Solution. Given that $\quad z(x+y)p + z(x-y)q = x^2 + y^2 \qquad \ldots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2} \qquad \ldots(2)$$

Choosing $x, -y, -z$ as multipliers, each fraction of (2) is

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2-y^2)} = \frac{x\,dx - y\,dy - zdz}{0}$$

$\therefore\ x\,dy - y\,dy - z\,dz = 0$ or $2x\,dx - 2y\,dy - 2z\,dz = 0$

Integrating it, we get $\quad x^2 - y^2 - z^2 = c_1 \qquad \ldots(3)$

where c_1 is an arbitrary constant.

Again, choosing $y, x, -z$ as multipliers, each fraction of (2) is

$$= \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}$$

$\therefore\ y\,dx + x\,dy - z\,dz = 0$ or $2d(xy) - 2z\,dz = 0$

Integrating it, we get $2xy - z^2 = c_2$ ...(4)

where c_2 is an arbitrary constant.

From (3) and (4), the required general solution is given by

$$\phi(x^2 - y^2 - z^2, 2xy - z^2) = 0 \quad \text{...(5)}$$

where ϕ is an arbitrary function.

Example 3. Solve $(mz - ny)p + (nx - lz)q = ly - mx.$

Solution. Given that $(mz - ny)p + (nx - lz)q = ly - mx$ ...(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \quad \text{...(2)}$$

Choosing x, y, z as multipliers, each fraction of (2) is

$$= \frac{x\,dx+ydy+zdx}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = \frac{xdx+ydy+zdz}{0}$$

$\therefore$ $xdx + ydy + zdx = 0$ or $2xdx + 2ydy + 2zdz = 0$

Integrating it, we get $x^2 + y^2 + z^2 = c_1$ ...(3)

where c_1 is an arbitrary constant.

Again, choosing l, m, n as multipliers, each fraction of (2) is

$$= \frac{ldx+mdy+ndz}{l(mx-ny)+m(nx-lz)+n(ly-mx)} = \frac{ldx+mdy+ndz}{0}$$

$\therefore$ We have $ldx + mdy + ndz = 0$

Integrating it, we get $lx + my + nz = c_2$ ...(4)

where c_2 is an arbitrary constant.

From (3) and (4), the required general solution is given by

$$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0 \qquad \ldots(5)$$

where ϕ is an arbitrary function.

Example 4. Solve $x(y^2 - z^2)q - y(z^2 + x^2)q = z(x^2 + y^2)$.

Solution. Given that $x(y^2 - z^2)q - y(z^2 + x^2)q = z(x^2 + y^2)$ …(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{-y(z^2+x^2)} = \frac{dz}{z(x^2+y^2)} \qquad \ldots(2)$$

Choosing $x, y, z,$ as multipliers, each fraction of (2) is

$$= \frac{xdx+ydy+zdz}{x^2(y^2-z^2)-y^2(z^2+x^2)+z^2(x^2+y^2)} = \frac{xdx+ydy+zdz}{0}$$

∴ We have $$xdx + ydy + zdz = 0$$

Integrating it, we get $$x^2 + y^2 + z^2 = c_1 \qquad \ldots(3)$$

Choosing $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ as multipliers, each fraction of (2) is

$$= \frac{(1/x)dx-(1/y)dy-(1/z)dz}{y^2-z^2+z^2+x^2-(x^2+y^2)} = \frac{(1/x)dx-(1/y)dy-(1/z)dz}{0}$$

∴ We have $(1/x)dx - (1/y)dy - (1/z)dz = 0$

Integrating it, we get $\log x - \log y - \log z = \log c_2$

or $\log\{x/(yz)\} = \log c_2$ or $x/yz = c_2$ …(4)

Using (3) and (4), the required general solution is given by

$$\phi\left(x^2 + y^2 + z^2, \frac{x}{yz}\right) = 0 \qquad \ldots(5)$$

where ϕ is an arbitrary function.

Example 5. Solve $(x-y)p+(x+y)q=2xz.$

Solution. Given that $(x-y)p+(x+y)q=2xz$ ...(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x-y}=\frac{dy}{x+y}=\frac{dz}{2xz} \quad \text{...(2)}$$

Taking the first two fractions of (2), we have

$$\frac{dy}{dx}=\frac{x+y}{x-y}=\frac{1+(y/x)}{1-(y/x)} \quad \text{...(3)}$$

which is a homogeneous differential equation in *x* and *y*.

Therefore, let us put $y/x=v$ i.e. $y=xv$ ...(4)

so that, we have $(dy/dx)=v+x(dv/dx)$ ...(5)

Using (4) and (5) in (2), we get $v+x\frac{dv}{dx}=\frac{1+v}{1-v}$

or $$x\frac{dv}{dx}=\frac{1+v}{1-v}-v=\frac{1+v-v(1-v)}{1-v}=\frac{1+v^2}{1-v}$$

Separating the variables, we can write

$$\frac{1-v}{1+v^2}dv=\frac{dx}{x} \quad \text{or} \quad \left(\frac{2}{1+v^2}-\frac{2v}{1+v^2}\right)dv=\frac{2dx}{x} \quad \text{...(6)}$$

Integrating it, we get $2\tan^{-1}v-\log(1+v^2)=2\log x-\log c_1$

or $$\log x^2-\log(1+v^2)-\log c_1=2\tan^{-1}v$$

or $\log\{x^2(1+v^2)/c_1\}=2\tan^{-1}v$ or $x^2(1+v^2)=c_1e^{2\tan^{-1}v}$

or $x^2[1+(y^2/x^2)]=c_1e^{2\tan^{-1}(y/x)}$, as $v=y/x$ by (4)

or $$(x^2+y^2)e^{-2\tan^{-1}\left(\frac{y}{x}\right)}=c_1 \quad \text{...(7)}$$

where c_1 is an arbitrary constant.

Choosing $1,1,-1/z$ as multipliers, each fraction of (2) is

$$= \frac{dx+dy-(1/z)dz}{(x-y)+(x+y)-(1/z)(2xz)} = \frac{dx+dy-(1/z)dz}{0}$$

$\therefore$ We have $$dx + dy - (1/z)dz = 0$$

Integrating it, we get $$x + y - \log z = c_2 \qquad \ldots(8)$$

where c_2 is an arbitrary constant.

From (7) and (8), the required general solution is given by

$$\phi\left((x^2+y^2)e^{-2\tan^{-1}(y/x)}, x + y - \log z\right) = 0 \qquad \ldots(9)$$

where ϕ is an arbitrary function.

EXERCISE 2(C)

Solve the following partial differential equations by Lagrange's method:

1. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$

2. $z(xp - yq) = y^2 - x^2$ **3.** $(y^2 + z^2)p - xyq + xz = 0$

4. $yp - xq = 2x - 3y$ **5.** $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$

ANSWERS

1. $\phi(x^2 + y^2 + z^2, xyz) = 0$ **2.** $\phi(x^2 + y^2 + z^2, xy) = 0$

3. $\phi(x^2 + y^2 + z^2, y/z) = 0$ **4.** $\phi(x^2 + y^2, 3x + 2y + z) = 0$

5. $\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0$

2.9.4 Rule IV for Solving $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Let the Lagrange's auxiliary equations for the PDE

$$Pp + Qq = R \qquad \ldots(1)$$

be
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots(2)$$

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well-known principle of algebra, each fraction of (2) will be equal to

$$(P_1dx + Q_1dy + R_1dz)/(P_1P + Q_1Q + R_1R) \qquad \ldots(3)$$

Suppose that the numerator of (3) is an exact differential of the denominator of (3). Then (3) can be combined with a suitable fraction in (2) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$(P_2dx + Q_2dy + R_2dz)/(P_2P + Q_2Q + R_2R) \qquad \ldots(4)$$

is such that its numerator is an exact differential of denominator. Fractions (3) and (4) are then combined to give an integral. This method may be repeated in some problems to get another integral. Sometimes, only one integral is possible by using the rule IV. In such cases, the second integral should be obtained by using rule 1 or rule 2 or rule 3 of previous articles.

The following solved examples will illustrate the rule:

SOLVED EXAMPLES

Example 1. Solve $(y + z)p + (z + x)q = x + y$.

Solution. Given that $(y + z)p + (z + x)q = x + y \qquad \ldots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad \ldots(2)$$

Choosing $1, -1, 0$ as multipliers, each fraction of (2) is

$$= \frac{dx-dy}{(y+z)-(z+x)} = \frac{d(x-y)}{-(x-y)} \qquad \ldots(3)$$

Again, choosing 0, 1, –1 as multipliers, each fraction of (2) is

$$= \frac{dy-dz}{(z+x)-(x+y)} = \frac{d(y-z)}{-(y-z)} \qquad \ldots(4)$$

Finally, choosing 1, 1, 1 as multipliers, each fraction of (2) is

$$= \frac{dx+dy+dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)} \qquad \ldots(5)$$

Now, from the fractions (3), (4) and (5), we get

$$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)} \qquad \ldots(6)$$

Taking the first two fraction of (6), we get $\frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$

Integrating it, we get $\log(x-y) = \log(y-z) + \log c_1$

or $\log\{(x-y)/(y-z)\} = \log c_1$ or $(x-y)/(y-z) = c_1$ …(7)

Taking the first and the third fractions of (6), we have

$$2\frac{d(x-y)}{(x-y)} + \frac{d(x+y+z)}{x+y+z} = 0$$

Integrating it, we get $2\log(x-y) + \log(x+y+z) = \log c_2$

or $$(x-y)^2(x+y+z) = c_2 \qquad \ldots(8)$$

From (7) and (8), the required general solution is given by

$$\phi[(x-y)/(y-z), (x-y)^2(x+y+z)] = 0 \qquad \ldots(9)$$

where ϕ is an arbitrary function.

Example 2. Solve $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$.

Solution. Given that $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$ …(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dz}{z(x^2+y^2)} \quad \ldots(2)$$

Taking the first two fractions of (1), we get

$$x^2dx = -y^2dy \quad \text{or} \quad 3x^2dx + 3y^2dy = 0$$

Integrating it, we get $x^3 + y^3 = c_1$ …(3)

Choosing $1, -1, 0$ as multipliers, each fraction of (2) is

$$= \frac{dx-dy}{y^2(x-y)+x^2(x-y)} = \frac{dx-dy}{(x-y)(x^2+y^2)} \quad \ldots(4)$$

Combining the third fraction of (2) with fraction (4), we get

$$\frac{dx-dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0$$

Integrating it, we get $\log(x-y) - \log z = \log c_2$

or $$\log\left\{\frac{(x-y)}{z}\right\} = \log c_2 \quad \text{or} \quad (x-y)/z = c_2 \quad \ldots(5)$$

From (4) and (5), the required solution is given by

$$\phi[x^3+y^3, (x-y)/z] = 0 \quad \ldots(6)$$

where ϕ is an arbitrary function.

Example 3. Solve $(x^2-y^2-yz)p + (x^2-y^2-zx)q = z(x-y)$

Solution. Given partial differential equation is

$$(x^2-y^2-yz)p + (x^2-y^2-zx)q = z(x-y) \quad \ldots(1)$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x^2-y^2-yz}=\frac{dy}{x^2-y^2-zx}=\frac{dz}{z(x-y)} \qquad \ldots(2)$$

Choosing $1,-1,0$ as multipliers, each fraction of (2) is

$$=\frac{dx-dy}{(x^2-y^2-yz)-(x^2-y^2-zx)}=\frac{dx-dy}{z(x-y)} \qquad \ldots(3)$$

Choosing $x,-y,0$ as multipliers, each fraction of (2) is

$$=\frac{xdx-ydy}{x(x^2-y^2-yz)-(x^2-y^2-zx)}=\frac{xdx-ydy}{(x-y)(x^2-y^2)} \qquad \ldots(4)$$

From the last fractions of (2), (3) and (4), we have

$$\frac{dx}{z(x-y)}=\frac{dx-dy}{z(x-y)}=\frac{xdx-ydy}{(x-y)(x^2-y^2)} \text{ or } \frac{dz}{z}=\frac{dx-dy}{z}=\frac{2xdx-2ydy}{2(x^2-y^2)} \qquad \ldots(5)$$

Taking the first two fractions of (5), we have

$$dz=dx-dy \quad \text{so that} \quad z-x+y=c_1 \qquad \ldots(6)$$

Again, taking the first and third fractions of (5), we have

$$d(x^2-y^2)/(x^2-y^2)-(2/z)dz=0 \qquad \ldots(7)$$

Integrating it, we get $\log(x^2-y^2)-2\log z=\log c_2$

or $$\log\left(\frac{x^2-y^2}{z^2}\right)=\log c_2 \text{ or } (x^2-y^2)/z^2=c_2 \qquad \ldots(8)$$

From (6) and (8), the required solution is given by

$$\phi[z-x+y,(x^2-y^2)/z^2]=0 \qquad \ldots(9)$$

where ϕ is an arbitrary function.

Example 4. Solve $\cos(x+y)\,p+\sin(x+y)\,q=z$.

Solution. Given that $\cos(x+y)\,p+\sin(x+y)\,q=z \qquad \ldots(1)$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} \qquad \ldots(2)$$

Choosing 1,1,0 as multipliers, each fraction of (2) is equal to

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} \qquad \ldots(3)$$

Choosing $1, -1, 0$ as multipliers, each fraction of (2) is

$$= \frac{dx-dy}{\cos(x+y)-\sin(x+y)} = \frac{d(x-y)}{\cos(x+y)-\sin(x+y)} \qquad \ldots(4)$$

From the last fractions of (2), (3) and from (4), we get

$$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{d(x-y)}{\cos(x+y)-\sin(x+y)} \qquad \ldots(5)$$

Taking the first two fractions of (5), we have

$$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} \qquad \ldots(6)$$

Putting $x+y=t$ so that $d(x+y)=dt$, (6) reduces to

$$\frac{dz}{z} = \frac{dt}{\cos t+\sin t} = \frac{dt}{\sqrt{2}\left\{\left(\frac{1}{\sqrt{2}}\right)\cos t+\left(\frac{1}{\sqrt{2}}\right)\sin t\right\}}$$

$$= \frac{dt}{\sqrt{2}\left\{\sin\left(\frac{\pi}{4}\right)\cos t+\cos\left(\frac{\pi}{4}\right)\sin t\right\}} = \frac{dt}{\sqrt{2}\sin(t+\pi/4)}$$

Thus, we have $\left(\sqrt{2}/z\right)dz = cosec\ (t+\pi/4)dt$

Integrating it, we get $\sqrt{2}\log z = \log\tan\frac{1}{2}\left(t+\frac{\pi}{4}\right) + \log c_1$

or $z^{\sqrt{2}} = c_1\tan\left(\frac{t}{2}+\frac{\pi}{8}\right)$ or $z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right) = c_1$, as $t = x+y$ ...(7)

Taking the last two fractions of (5), we have

$$d(x-y) = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x+y) \qquad \ldots(8)$$

Putting $x + y = t$ so that $d(x + y) = dt$, (7) reduces to

$$d(x - y) = \frac{\cos t - \sin t}{\cos t + \sin t} dt \quad \text{so that} \quad x - y = \log(\sin t + \cos t) - \log c_2$$

or $\quad (\sin t + \cos t)/c_2 = e^{x-y} \quad$ or $\quad e^{-(x-y)}(\sin t + \cos t) = c_2$

or $\quad e^{y-x}[\sin(x + y) + \cos(x + y)] = c_2$, as $t = x + y$ $\quad$ ...(9)

From (7) and (9), the required general solution is given by

$$\phi \left[z^{\sqrt{2}} \cot\left(\frac{x+y}{2} + \frac{\pi}{8}\right), e^{y-x}\{\sin(x + y) + \cos(x + y)\} \right] = 0 \quad \ldots(10)$$

where ϕ is an arbitrary function.

Example 5. Solve $(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2z(x^2 + y^2)$.

Solution. Given partial differential equation is

$$(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2z(x^2 + y^2) \quad \ldots(1)$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x^3+3xy^2} = \frac{dy}{y^3+3x^2y} = \frac{dz}{2z(x^2+y^2)} \quad \ldots(2)$$

Choosing 1,1,0 as multipliers, each fraction of (2) is

$$= \frac{dx+dy}{x^3+3xy^2+3x^2y+y^3} = \frac{d(x+y)}{(x+y)^3} \quad \ldots(3)$$

Choosing $1, -1, 0$ as multipliers, each fraction of (2) is

$$= \frac{dx-dy}{x^3+3xy^2-y^3-3x^2y} = \frac{d(x-y)}{(x-y)^3} \quad \ldots(4)$$

From the last fractions of (3) and (4), we get

$$(x + y)^{-3} d(x + y) = (x - y)^{-3} d(x - y)$$

or $\quad u^{-3} du - v^{-3} dv = 0$, on putting $u = x + y$ and $v = x - y$

Integrating it, we get $v^{-2} - u^{-2} = c_1$

or $(x-y)^{-2} - (x+y)^{-2} = c_1$, as $u = x + y$ and $v = x - y$ ...(5)

Choosing $1/x, 1/y, 0$ as multipliers, each fraction of (2) is

$$= \frac{(1/x)dx+(1/y)dy}{(1/x)\times(x^3+3xy^2)+(1/y)\times(y^3+3x^2y)} = \frac{(1/x)dx+(1/y)dy}{4(x^2+y^2)} \quad ...(6)$$

Combining the last fraction of (2) with fraction (6), we have

$$\frac{dz}{2z(x^2+y^2)} = \frac{(1/x)dx+(1/y)dy}{4(x^2+y^2)} \quad \text{or} \quad \frac{dx}{x} + \frac{dy}{y} - 2\frac{dz}{z} = 0$$

Integrating it, we get $\log x + \log y - 2\log z = \log c_2$

or $$\log\left(\frac{xy}{z^2}\right) = \log c_2 \quad \text{or} \quad xy/z^2 = c_2 \quad ...(7)$$

From (5) and (7), the required solution is given by

$$\phi[(x-y)^{-2} - (x+y)^{-2}, (xy)/z^2] = 0 \quad ...(8)$$

where ϕ is an arbitrary function.

Example 6. Solve $p + q = x + y + z.$

Solution. Given that $p + q = x + y + z$ ...(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x+y+z} \quad ...(2)$$

Taking the first two fractions of (2), we get

$$dx - dy = 0 \quad \text{so that} \quad x - y = c_1 \quad ...(3)$$

Choosing 1,1,1 as multipliers, each fraction of (2) is

$$= \frac{dx+dy+dz}{1+1+(x+y+z)} = \frac{d(2+x+y+z)}{2+x+y+z} \quad ...(4)$$

Combining first fraction of (2) with second fraction of (4), we get

$$d(2+x+y+z)/(2+x+y+z) = dx$$

Integrating it, we have $\log(2+x+y+z) - \log c_2 = x$

or $(2+x+y+z)/c_2 = e^x$ or $(2+x+y+z)e^{-x} = c_2$ …(5)

From (3) and (5), the required general solution is given by

$$\phi[x-y, (2+x+y+z)e^{-x}] = 0 \qquad \text{…(6)}$$

where ϕ is an arbitrary function.

Example 7. Solve the PDE $(2x^2+y^2+z^2-2yz-zx-xy)p + (x^2+2y^2+z^2-yz-2zx-xy)q = x^2+y^2+2z^2-yz-zx-2xy$.

Solution. Given partial differential equation can be written as

$$Pp + Qq = R \qquad \text{…(1)}$$

where $P = 2x^2+y^2+z^2-2yz-zx-xy$,

$Q = x^2+2y^2+z^2-yz-2zx-xy$,

and $R = x^2+y^2+2z^2-yz-zx-2xy$

The Lagrange's auxiliary equations for (1) are

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, which can also be written as given below:

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy} = \frac{dy}{x^2+2y^2+z^2-yz-2zx-xy} = \frac{dz}{x^2+y^2+2z^2-yz-zx-2xy}$$

…(2)

Choosing $1,-1,0; 0,1,-1$ and $-1,0,1$ as multipliers in turn, each fraction of (2) is equal to

$$= \frac{dx-dy}{x^2-y^2-yz+zx} = \frac{dy-dz}{y^2-z^2-zx+xy} = \frac{dz-dx}{z^2-x^2-xy+yz}$$

or $$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)} = \frac{dz-dx}{(z-x)(x+y+z)} \qquad \ldots(3)$$

Taking the first two fractions of (3), we have

$$\frac{d(x-y)}{(x-y)} - \frac{d(y-z)}{(y-z)} = 0$$

Integrating it, we get $\log(x-y) - \log(y-z) = \log c_1$

or $$(x-y)/(y-z) = c_1 \qquad \ldots(4)$$

Taking the last two fractions of (3), we get

$$\frac{d(y-z)}{(y-z)} - \frac{d(z-x)}{(z-x)} = 0$$

Integrating it, we get $\log(y-z) - \log(z-x) = \log c_2$

or $$(y-z)/(z-x) = c_2 \qquad \ldots(5)$$

From (4) and (5), the required general solution is given by

$$\phi[(x-y)/(y-z), (y-z)/(z-x)] = 0 \qquad \ldots(6)$$

where ϕ is an arbitrary function.

Example 8. Solve the following partial differential equation:

$$\{my(x+y) - nz^2\}\left(\frac{\partial z}{\partial x}\right) - \{lx(x+y) - nz^2\}\left(\frac{\partial z}{\partial y}\right) = (lx - my)z$$

Solution. The given partial differential equation may be written as

$$\{my(x+y) - nz^2\}p - \{lx(x+y) - nz^2\}q = (lx - my)z \quad \ldots(1)$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{my(x+y)-nz^2} = \frac{dy}{-lx(x+y)+nz^2} = \frac{dz}{(lx-my)z} \qquad \ldots(2)$$

Taking 1, 1 and 0 as multipliers, each fraction of (2) is

$$= \frac{dx+dy}{my(x+y)-lx(x+y)} = \frac{dx+dy}{(my-lx)(x+y)} \qquad \ldots(3)$$

Now, from the last fractions of (2) and (3), we have

$$\frac{dx+dy}{(my-lx)(x+y)} = \frac{dz}{-(my-lx)z} \text{ or } \frac{d(x+y)}{x+y} = -\frac{dz}{z}$$

Integrating it, we get $\log(x+y) = -\log z + \log c_1$

or $(x+y)z = c_1$ or $xy + yz = c_1$ ...(4)

Taking lx, my, nz as multipliers, each fraction of (2) is

$$\frac{lxdx+mydy+nzdz}{lx\,my(x+y)-lxnz^2-my\,lx(x+y)+my\,nz^2+nz^2(lx-my)} = \frac{lx\,dx+my\,dy+nz\,dz}{0}$$

∴ $2lx\,dx + 2my\,dy + 2nz\,dz = 0$ so that $lx^2 + my^2 + nz^2 = c_2$ ...(5)

From (4) and (5), the required solution is given by

$$\phi(xy + yz, lx^2 + my^2 + nz^2) = 0 \qquad \ldots(6)$$

where ϕ is an arbitrary function.

EXERCISE 2(D)

Solve the following partial differential equations:

1. $(y^2 + yz + z^2)p + (z^2 + zx + x^2)q = x^2 + xy + y^2$
2. $x^2p + y^2q = (x+y)z$
3. $x(z - 2y^2) = (z - yq)(z - y^2 - 2x^3)$
4. $(x^2 + y^2)p + 2xyq = (z + y)$
5. $\{y(x+y) + az\}p + \{x(x+y) - az\}q = z(x+y)$

ANSWERS

1. $\phi\left(\frac{y-z}{x-y}, \frac{x-z}{x-y}\right) = 0$ **2.** $\phi\left(\frac{xy}{z}, \frac{x-y}{z}\right) = 0$ **3.** $\phi\left(\frac{y}{z}, \frac{z-y^2+x^3}{x}\right) = 0$

4. $\phi\left(\frac{x+y}{z}, \frac{y}{x^2-y^2}\right) = 0$ **5.** $\phi\left(\frac{x+y}{z}, x^2 - y^2 - 2az\right)$

2.10 Surfaces and Normals in Three Dimensions

Let Ω be a domain in three-dimensional space R^3 and let $\phi(x, y, z)$ be a scalar point function, then the vector valued function grad ϕ may be written as

$$\text{grad } \phi = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \qquad \ldots(1)$$

If we assume that the partial derivatives of ϕ do not vanish simultaneously at any point, then the set of points (x, y, z) in Ω, satisfying the equation

$$\phi(x, y, z) = C \qquad \ldots(2)$$

is a surface in Ω for some constant C. This surface is called a **level** or **equipotential surface of** ϕ. If (x_0, y_0, z_0) is a given point in Ω, then by taking $\phi(x_0, y_0, z_0) = C$, we get an equation of the form

$$\phi(x, y, z) = \phi(x_0, y_0, z_0) \qquad \ldots(3)$$

which represents a surface in the domain Ω of three dimensional space passing through the point (x_0, y_0, z_0). Here, equation (2) represents a one-parameter family of surface in the domain Ω. The value of grad ϕ is a vector, normal to the level surface. Now, one may ask, if it is possible to solve equation (2) for *z* in terms of *x* and *y*. To answer this question, let us consider a set of relations of the form

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v) \qquad \ldots(4)$$

Here, for every pair of values of *u* and *v*, we will have three numbers *x*, *y* and *z*, which represent a point in space. However, it may be noted that every point in space need not correspond to a pair (*u*, *v*). But if the Jacobian

$$\frac{\partial(f_1,f_2)}{\partial(u,v)} \neq 0 \qquad \ldots(5)$$

then, the first two equations of (4) can be solved and *u* and *v* can be expressed as functions of *x* and *y* like

$$u = \lambda(x, y) \text{ and } v = \mu(x, y) \qquad \ldots(6)$$

Thus, the third relation of equation (4) gives the value of z in the form

$$z = f_3[\lambda(x, y), \mu(x, y)] \qquad \ldots(7)$$

This relation is of course, a functional relation between the co-ordinates *x*, *y* and *z* as in equation (2). Hence, any point (*x*, *y*, *z*) obtained from equation (4) always lie on a fixed surface. The set of equations (4) are called as the **parametric equations of a surface.** It may be noted that the parametric equations of a surface need not be unique, which can be seen in the following example:

The following two sets of parametric equations

$$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, z = r\cos\theta \qquad \textbf{(set I)}$$

and $$x = r\frac{(1-\phi^2)}{(1+\phi^2)}\cos\theta, \quad y = r\frac{(1-\phi^2)}{(1+\phi^2)}\sin\theta, z = \frac{2r\phi}{1+\phi^2} \qquad \textbf{(set II)}$$

represent the same surface $x^2 + y^2 + z^2 = r^2$, which is a sphere.

Now, let us take the surface whose equation is of the form

$$z = f(x, y) \qquad \ldots(8)$$

The above equation may also be written in the form

$$\phi \equiv f(x, y) - z = 0 \qquad \ldots(9)$$

Differentiating it partially w.r.t. *x* and *y*, we obtain

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial z}\frac{\partial z}{\partial x}=0 \text{ and } \frac{\partial \phi}{\partial y}+\frac{\partial \phi}{\partial z}\frac{\partial z}{\partial y}=0 \qquad \ldots(10)$$

form which on using (9), we get $\frac{\partial z}{\partial x}=-\frac{\partial \phi/\partial x}{\partial \phi/\partial z}=\frac{\partial \phi}{\partial x}$ i.e., $\frac{\partial \phi}{\partial x}=p$

Thus, we have $$\frac{\partial \phi}{\partial x}=p, \frac{\partial \phi}{\partial y}=q, \frac{\partial \phi}{\partial z}=-1 \qquad \ldots(11)$$

Hence, the direction cosines of the normal to the surface at a point P (x ,y, z) are given by

$$\left(\frac{p}{\sqrt{p^2+q^2+1}}, \frac{q}{\sqrt{p^2+q^2+1}}, \frac{-1}{\sqrt{p^2+q^2+1}}\right) \qquad \ldots(12)$$

Now, returning to the level surface given by equation (2), it is easy to write the equation of the tangent plane to the level surface at a point (x_0, y_0, z_0) as

$$(x-x_0)\left[\frac{\partial F}{\partial x}\right]_{(x_0,y_0,z_0)}+(y-y_0)\left[\frac{\partial F}{\partial y}\right]_{(x_0,y_0,z_0)}+(z-z_0)\left[\frac{\partial F}{\partial z}\right]_{(x_0,y_0,z_0)}=0$$

2.11 Curve in Three Dimensions: Intersection of Two Surfaces

A curve in three-dimensional space R^3 can be described in terms of parametric equations. Suppose $\vec{r}$ denotes the position vector of a point on a curve C, then the vector equation of the curve C may be written as

$$\vec{r}=\vec{F}(t) \quad , \quad t \in I \qquad \ldots(1)$$

where I is some interval on the real axis. In component form, equation (1) can be written as

$$x=f_1(t),\ y=f_2(t),\ z=f_3(t) \qquad \ldots(2)$$

where $\vec{r}=(x,y,z)$ and $\vec{F}(t)=[f_1(t), f_2(t), f_3(t)\,]$.

Further, we assume that

$$\left(\frac{df_1(t)}{dt}, \frac{df_2(t)}{dt}, \frac{df_3(t)}{dt}\right) \neq (0,0,0) \qquad \ldots(3)$$

This non-vanishing vector is called as the **tangent vector** to the curve C at the point (x ,y, z) or at $[f_1(t), f_2(t), f_3(t)\,]$ to the curve C.

Another way of describing a curve in three-dimensional space R^3 is by using the fact that the intersection of the surfaces gives rise to a curve.

$$\text{Let} \quad \phi_1(x,y,z) = C_1 \quad \text{and} \quad \phi_2(x,y,z) = C_2 \qquad \ldots(4)$$

are two surfaces. Their intersection, if not empty, is always a curve, provided grad ϕ_1 and grad ϕ_2 are not collinear at any point of the domain Ω . In other words, the intersection of surfaces given by equation (4) is a curve if

$$\text{grad } \phi_1(x,y,z).\,\text{grad } \phi_2(x,y,z) \neq (0,0,0) \qquad \ldots(5)$$

for every (x, y, z) ∈ Ω. For various values of C_1 and C_2, equation (4) describes different curves. The totality of these curves is called a **two parameter family of curves**. Here, C_1 and C_2 are referred to as parameters of this family.

2.12 Tangent Line: Intersection of Two Tangent Planes

Let us consider two surfaces denoted by S_1 and S_2 whose equations are given by

$$F(x,y,z) = 0 \qquad \ldots(1)$$

and

$$G(x,y,z) = 0 \qquad \ldots(2)$$

Then, the equation of the tangent plane π_1 to the surface S_1 at a point $P(x_0, y_0, z_0)$ is given by

$$(x - x_0)\frac{\partial F}{\partial x} + (y - y_0)\frac{\partial F}{\partial y} + (z - z_0)\frac{\partial F}{\partial z} = 0 \qquad \ldots(3)$$

where $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$ are all evaluated at the point $P(x_0, y_0, z_0)$.

Similarly, the equation of the tangent plane π_2 to the surface S_2 at the point $P(x_0, y_0, z_0)$ is given by

$$(x - x_0)\frac{\partial G}{\partial x} + (y - y_0)\frac{\partial G}{\partial y} + (z - z_0)\frac{\partial G}{\partial z} = 0 \qquad \ldots(4)$$

where the partial derivatives $\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}$ and $\frac{\partial G}{\partial z}$ are all evaluated at the point $P(x_0, y_0, z_0)$.

The intersection of tangent planes is known as the tangent line at $P(x_0, y_0, z_0)$. Thus, the tangent line L to the curve C at the point $P(x_0, y_0, z_0)$ is the intersection of the two surfaces S_1 and S_2.

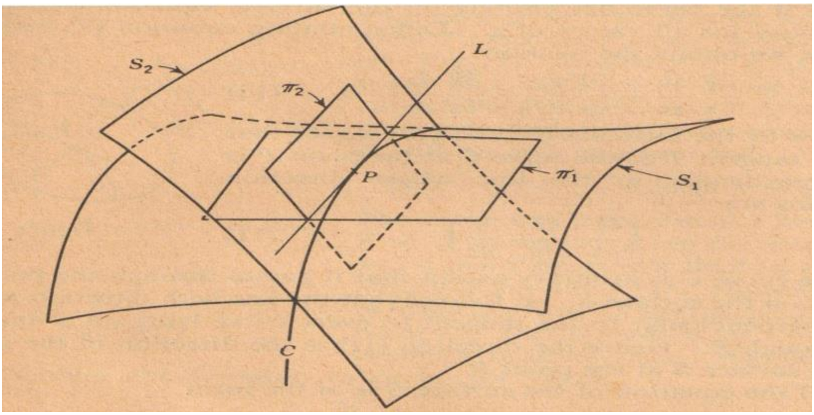

Figure 2.2: Tangent Planes and Tangent Line

The equation of the tangent line L to the curve C at the point $P(x_0, y_0, z_0)$ is obtained from (3) and (4), which is given below:

$$\frac{(x-x_0)}{\frac{\partial F}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}\frac{\partial G}{\partial y}} = \frac{(y-y_0)}{\frac{\partial F}{\partial z}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z}} = \frac{(z-z_0)}{\frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x}}$$

or

$$\frac{(x-x_0)}{\frac{\partial(F,G)}{\partial(y,z)}} = \frac{(y-y_0)}{\frac{\partial(F,G)}{\partial(z,x)}} = \frac{(z-z_0)}{\frac{\partial(F,G)}{\partial(x,y)}} \qquad \ldots(5)$$

Therefore, the direction cosines of the tangent line L are proportional to $\left[\frac{\partial(F,G)}{\partial(y,z)}, \frac{\partial(F,G)}{\partial(z,x)}, \frac{\partial(F,G)}{\partial(x,y)}\right]$ ...(6)

2.13 Integral Surfaces Passing Through a Given Curve

In the previous article, we have obtained general integral of the partial differential equation $Pp + Qq = R$.

We shall now present two methods for finding the integral surface which passes through a given curve

2.13.1 First Method for Finding Integral Surface

Let $$Pp + Qq = R \qquad \ldots(1)$$

be the given PDE. Let its Lagrange's auxiliary equations give us the following two independent solutions

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \qquad \ldots(2)$$

Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by

$$x = x(t), \; y = y(t), \text{ and } \; z = z(t) \qquad \ldots(3)$$

where t is a parameter. Then (2) may be expressed as

$$u[x(t), y(t), z(t)] = c_1 \;\text{ and }\; v[x(t), y(t), z(t)] = c_2 \qquad \ldots(4)$$

We now eliminate the parameter t from the equations of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (2) and obtain the required integral surface.

2.13.2 Second Method for Finding Integral Surface

Let $$Pp + Qq = R \qquad \ldots(1)$$

be the given PDE. Let its Lagrange's auxiliary equations give us the following two independent integrals:

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \qquad \ldots(2)$$

Suppose we wish to obtain the integral surface passing through the curve which is determined by the following two equations:

$$\phi(x, y, z) = 0 \quad \text{and} \quad \psi(x, y, z) = 0 \qquad \ldots(3)$$

We now eliminate x, y and z from the two pairs of equations of (2) and (3) and obtain a relation between c_1 and c_2. Finally, we replace c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ in that relation and we obtain the desired integral surface.

2.14 Surfaces Orthogonal to a Given System of Surfaces

Let $$f(x, y, z) = c \qquad \ldots(1)$$

represents a system or surfaces, where c is a parameter. Suppose we wish to obtain a system of surfaces which cut each of (1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to (1) which passes through that point are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

Let the surface $$z = \phi(x, y) \qquad \ldots(2)$$

cuts each surface of (1) at right angles. Then the normal at (x, y, z) to (2) has direction ratios $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1$ i.e., $p, q, -1$. Since normals at $P(x, y, z)$ to (1) and (2) are at right angles, therefore, we have

$$p\left(\frac{\partial f}{\partial x}\right) + q\left(\frac{\partial f}{\partial y}\right) - \left(\frac{\partial f}{\partial z}\right) = 0 \quad \text{or} \quad p\left(\frac{\partial f}{\partial x}\right) + q\left(\frac{\partial f}{\partial y}\right) = \left(\frac{\partial f}{\partial z}\right) \quad \ldots(3)$$

which is of the form $Pp + Qq = R$, where $P = \frac{\partial f}{\partial x}, Q = \frac{\partial f}{\partial y}$ and $R = \frac{\partial f}{\partial z}$.

Conversely, we may easily verify that any solution of (3) is orthogonal to every surface of (1).

2.15 Geometrical Description of Solutions of Lagrange's Equation $\boldsymbol{Pp + Qq = R}$ and Lagrange's Auxiliary Equations $\boldsymbol{\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{R}}$

Proof. Consider $$Pp + Qq = R \qquad \ldots(1)$$

and $$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots(2)$$

where P, Q and R are functions of *x*, *y* and *z*.

Let $$z = \phi(x, y) \qquad \ldots(3)$$

represents the solution of the Lagrange's partial differential equation (1). Then (3) represents a surface whose normal at any point (x, y, z) has direction ratios $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1$ i.e., $p, q, -1$. Also, we know that the system of simultaneous equations (2) represent a family of curves such that the tangent at any point has direction ratios P, Q, R. Rewriting (1), we have

$$Pp + Qq + R(-1) = 0 \qquad \ldots(4)$$

which shows that the normal to the surface (3) at any point is perpendicular to the member of family of curves (2) through that point. Hence, the member must touch the surface at that point. Since this holds for each point on (3), therefore, we consider that the curves (2) lies completely on the surface (3) whose differential equation is given by (1).

2.16 Geometrical Interpretation of $\boldsymbol{Pp + Qq = R}$

Here, we show that the surface given by $Pp + Qq = R$ is orthogonal to the surfaces represented by $Pdx + Qdy + Rdz = 0$.

We know that the curves whose equations are solutions of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \ldots(1)$$

are orthogonal to the system of the surfaces satisfying

$$Pdx + Qdy + Rdz = 0 \qquad \ldots(2)$$

Again, the curves of (1) lie completely on the surface

$$Pp + Qq = R \qquad \ldots(3)$$

Hence, we conclude that surfaces represented by (2) and (3) are orthogonal.

SOLVED EXAMPLES

Example 1. Find the tangent vector at the point $\left(0,1,\frac{\pi}{2}\right)$ to the helix described by the parametric equations $x = \cos t\,, y = \sin t\,, z = t.$

Solution. The tangent vector to the helix at (x, y, z) is given by

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (-\sin t\,, \cos t\,, 1)$$

We observe that the given point $\left(0,1,\frac{\pi}{2}\right)$ corresponds to $t = \frac{\pi}{2}$.

Therefore, the required tangent vector to the helix is given by

$$\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \left(-\sin\frac{\pi}{2}, \cos\frac{\pi}{2}, 1\right) = (-1,0,1).$$

Example 2. Find the equation of the tangent line to the space circle

$x^2 + y^2 + z^2 = 1, x + y + z = 0$ at the point $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$.

Solution. The space circle is described by the functions:

$$F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \qquad \ldots(1)$$

and $$G(x, y, z) = x + y + z = 0 \qquad \ldots(2)$$

The equation of the tangent line at the point (x_0, y_0, z_0) is

$$\frac{(x-x_0)}{\frac{\partial(F,G)}{\partial(y,z)}} = \frac{(y-y_0)}{\frac{\partial(F,G)}{\partial(z,x)}} = \frac{(z-z_0)}{\frac{\partial(F,G)}{\partial(x,y)}} \qquad \ldots(3)$$

where $\frac{\partial(F,G)}{\partial(y,z)} = \frac{\partial F}{\partial y}\frac{\partial G}{\partial z} - \frac{\partial F}{\partial z}\frac{\partial G}{\partial y} = 2y - 2z = \frac{4}{\sqrt{14}} + \frac{6}{\sqrt{14}} = \frac{10}{\sqrt{14}}$

$$\frac{\partial(F,G)}{\partial(z,x)} = \frac{\partial F}{\partial z}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z} = 2z - 2x = -\frac{6}{\sqrt{14}} - \frac{2}{\sqrt{14}} = -\frac{8}{\sqrt{14}}$$

$$\frac{\partial(F,G)}{\partial(x,y)} = \frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial x} = 2x - 2y = \frac{2}{\sqrt{14}} - \frac{4}{\sqrt{14}} = -\frac{2}{\sqrt{14}}$$

The required equation of the tangent line at the given point $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$ is given by

$$\frac{x-1/\sqrt{14}}{10/\sqrt{14}} = \frac{y-2/\sqrt{14}}{-8/\sqrt{14}} = \frac{z+3/\sqrt{14}}{-2/\sqrt{14}} \qquad \ldots(4)$$

Example 3. Find the integral surface of the linear partial differential equation $x(x^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which contains the straight line $x+y=0, z=1$.

Solution. Given $x(x^2+z)p - y(x^2+z)q = (x^2-y^2)z \qquad \ldots(1)$

The Lagrange's auxiliary equations of (1) are

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \qquad \ldots(2)$$

The two independent solutions of (2) may be obtained as

$$u(x,y,z) = xyz = c_1 \qquad \ldots(3)$$

and $\qquad v(x,y,z) = x^2 + y^2 - 2z = c_2 \qquad \ldots(4)$

Taking t as parameter, the given equation of the straight line $x+y=0, z=1$ can be put in parametric form

$$x = t, \qquad y = -t, \qquad z = 1 \qquad \ldots(5)$$

Using (5) in (3) and (4), we get $-t^2 = c_1$ and $2t^2 - 2 = c_2$ ...(6)

Eliminating t from the equations of (6), we get

$$2(-c_1) - 2 = c_2 \quad \text{or} \quad 2c_1 + c_2 + 2 = 0 \qquad ...(7)$$

Now, putting the values of c_1 and c_2 from (3) and (4) in (7), we get

$$2xyz + x^2 + y^2 - 2z + 2 = 0 \qquad ...(8)$$

which is the desired integral surface of the given PDE.

Example 4. Find the equation of the integral surface of the partial differential equation $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z = 0, x^2 + y^2 = 2x$.

Solution. Given that $2y(z-3)p + (2x-z)q = y(2x-3)$ ...(1)

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \qquad ...(2)$$

Taking the first and third fractions of (3), we get

$$(2x-3)dx - 2(z-3)dz = 0$$

Integrating it, we get $x^2 - 3x - z^2 + 6z = c_1$ ...(3)

Choosing $\frac{1}{2}, y, -1$ as multipliers, each fraction of (2) is

$$= \frac{(1/2)dx + ydy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = \frac{(1/2)dx + ydy - dz}{0}$$

Hence $(1/2)dx + ydy - dz = 0$ or $dx + 2ydy - 2dz = 0$

Integrating it, we get $x + y^2 - 2z = c_2$ ...(4)

Thus, the two independent solutions of (2) are given by (3) and (4).

Now, the parametric equations of given circle are

$$x = t, \quad y = (2t - t^2)^{1/2}, z = 0 \qquad \ldots(5)$$

Substituting these values of x, y and z in (3) and (4), we have

$$t^2 - 3t = c_1 \quad \text{and} \quad 3t - t^2 = c_2 \qquad \ldots(6)$$

Eliminating t from the equations of (6), we get

$$c_1 + c_2 = 0 \qquad \ldots(7)$$

Substituting the values of c_1 and c_2 from (3) and (4) in (7), the desired integral surface is given by

$$x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$$

or $$x^2 + y^2 - z^2 - 2x + 4z = 0 \qquad \ldots(8)$$

Example 5. Find the integral surface of the partial differential equation $(x - y)p + (y - x - z)q = z$ passing through the circle $z = 1, x^2 + y^2 = 1$.

Solution. Given that $$(x - y)p + (y - x - z)q = z \qquad \ldots(1)$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{x-y} = \frac{dy}{y-x-z} = \frac{dz}{z} \qquad \ldots(2)$$

Choosing 1,1,1 as multipliers, each fraction of (2) is

$$= \frac{dx+dy+dz}{x-y+y-x-z+z} = \frac{dx+dy+dz}{0}$$

$\therefore \quad dx + dy + dz = 0$ so that $x + y + z = c_1 \qquad \ldots(3)$

Taking the last two fractions of (2) and using (3), we get

$$\frac{dy}{y-(c_1-y)} = \frac{dz}{z} \quad \text{or} \quad \frac{2dy}{2y-c_1} - \frac{2dz}{z} = 0$$

Integrating it, we get $\log(2y - c_1) - 2\log z = \log c_2$

$$\log\left(\frac{2y-c_1}{z^2}\right) = \log c_2 \ \text{ or } \ (2y - c_1)/z^2 = c_2$$

or $(2y - x - y - z)/z^2 = c_2$ or $(y - x - z)/z^2 = c_2$ ...(4)

The equation of the given curve (circle) is

$$z = 1, \qquad x^2 + y^2 = 1 \qquad \text{...(5)}$$

Putting $z = 1$ in (3) and (4), we get

$$x + y = c_1 - 1 \quad \text{and} \quad y - x = c_2 + 1 \qquad \text{...(6)}$$

But $2(x^2 + y^2) = (x + y)^2 + (x - y)^2$ ...(7)

Using equations (5) and (6) in (7), we get

$$2 = (c_1 - 1)^2 + (c_2 + 1)^2 \ \text{ or } \ c_1^2 + c_2^2 - 2c_1 + 2c_2 = 0 \qquad \text{...(8)}$$

Putting the values of c_1 and c_2 from (3) and (4) in (8), the required integral surface is given by

$$(x + y + z)^2 + (y - x - z)^2/z^4 - (x + y + z)$$
$$+2(y - x - z)/z^2 = 0$$

or
$$z^4(x + y + z)^2 + (y - x - z)^2$$
$$-2z^4(x + y + z) + 2z^2(y - x - z) = 0$$

Example 6. Find the equation of integral surface satisfying $4yzp + q + 2y = 0$ and passing through $y^2 + z^2 = 1, x + z = 2$.

Solution. Given that $4yzp + q = -2y$ ...(1)

The equation of the given curve is

$$y^2 + z^2 = 1, \quad x + z = 2 \qquad \ldots(2)$$

The Lagrange's auxiliary equations for (1) are

$$\frac{dx}{4yz} = \frac{dy}{1} = \frac{dz}{-2y} \qquad \ldots(3)$$

Taking the first and third fractions of (3), we have

$$dx + 2zdz = 0 \text{ so that } x + z^2 = c_1 \qquad \ldots(4)$$

Taking the last two fractions of (3), we have

$$dz + 2ydy = 0 \text{ so that } z + y^2 = c_2 \qquad \ldots(5)$$

Adding (4) and (5), we get $(y^2 + z^2) + (x + z) = c_1 + c_2$

or $$1 + 2 = c_1 + c_2 \text{, using (2)} \qquad \ldots(6)$$

Putting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required integral surface is given by

$$3 = x + z^2 + z + y^2 \quad \text{or} \quad y^2 + z^2 + x + z - 3 = 0$$

Example 7. Find the surface which intersects the surfaces of the system $z(x + y) = c(3z + 1)$ orthogonally and which passes through the circle $x^2 + y^2 = 1, z = 1$.

Solution. The equation of the given system of surfaces is

$$f(x, y, z) \equiv \frac{z(x+y)}{3z+1} = c \qquad \ldots(1)$$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{z}{3z+1}, \quad \frac{\partial f}{\partial y} = \frac{z}{3z+1}, \quad \frac{\partial f}{\partial z} = \left[\frac{(3z+1) - 3z}{(3z+1)^2}\right](x + y) = \frac{x+y}{(3z+1)^2}$$

The required orthogonal surface will be the solution of

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \quad \text{or} \quad \frac{zp}{3z+1} + \frac{zq}{3z+1} = \frac{x+y}{(3z+1)^2}$$

or $$z(3z+1)q + z(3z+1)q = x + y \qquad \ldots(2)$$

The Lagrange's auxiliary equations for (2) are

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y} \qquad \ldots(3)$$

Taking the first two fractions of (3), we get $dx - dy = 0$

Integrating it, we get $\quad x - y = c_1 \qquad \ldots(4)$

Taking $x, y, -z(3z+1)$ as multipliers, each fraction of (3) is

$$= [xdx + ydy - z(3z+1)dz]/0$$

$\therefore$ $$xdx + ydy - 3z^2dz - zdz = 0$$

or $$2xdx + 2ydy - 6z^2dz - 2zdz = 0$$

Integrating it, we get $\quad x^2 + y^2 - 2z^3 - z^2 = c_2 \qquad \ldots(5)$

Hence, the surface which is orthogonal to (1) is given by

$$x^2 + y^2 - 2z^3 - z^2 = \phi(x - y) \qquad \ldots(6)$$

where ϕ is an arbitrary function.

In order to get the desired surface passing through the circle $x^2 + y^2 = 1, z = 1$, we must choose $\phi(x - y) = -2$.

Thus, the required particular surface is given by

$$x^2 + y^2 - 2z^3 - z^2 = -2$$

Example 8. Write down the system of equations for obtaining the general equation of surfaces orthogonal to the family given by $x(x^2 + y^2 + z^2) = cy^2$.

Solution. The equation of the given family of surface is

$$f(x,y,z) \equiv x(x^2+y^2+z^2)/y^2 = c \qquad \dots(1)$$

The surfaces orthogonal to the system (1) are the surfaces generated by the integral curves of the equations

$$\frac{dx}{\frac{\partial f}{\partial x}} = \frac{dy}{\frac{\partial f}{\partial y}} = \frac{\partial z}{\frac{\partial f}{\partial z}} \text{ or } \frac{dx}{(3x^2+y^2+z^2)/y^2} = \frac{dy}{-2x(x^2+z^2)/y^3} = \frac{dz}{2x/y^2z}$$

or
$$\frac{dx}{y(3x^2+y^2+z^2)} = \frac{dy}{-2x(x^2+z^2)} = \frac{dz}{2xyz} \qquad \dots(2)$$

Taking x, y, z as multipliers, each fraction of (2) is

$$= \frac{xdx+ydy+zdz}{xy(3x^2+y^2+z^2)-2x(x^2+z^2)+2xyz} = \frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)} \qquad \dots(3)$$

Combining this fraction (3) with the last fraction of (2), we get

$$\frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)} = \frac{dz}{2xyz} \quad \text{or} \quad \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} = \frac{dz}{z}$$

Integrating it, we get $\log(x^2+y^2+z^2) = \log z + \log c_1$

or $\quad x^2+y^2+z^2 = c_1 z \quad$ or $\quad (x^2+y^2+z^2)/z = c_1 \qquad \dots(4)$

Taking $4x, 2y, 0$ as multipliers, each fraction of (2) is

$$\frac{4xax+2ydy}{4xy(3x^2+y^2+z^2)-4xy(x^2+y^2)} = \frac{4xd+2ydy}{4xy(2x^2+y^2)} \qquad \dots(5)$$

Combining this fraction (5) with the last fraction of (2), we get

$$\frac{4xdx+2ydy}{4xy(2x^2+y^2)} = \frac{dz}{2xyz} \quad \text{or} \quad \frac{4xdx+2ydy}{2x^2+y^2} = \frac{2dz}{z}$$

Integrating it, we get $\log(2x^2+y^2) = 2\log z + \log c_2$

or $\quad 2x^2+y^2 = c_2 z^2 \quad$ or $\quad (2x^2+y^2)/z^2 = c_2 \qquad \dots(6)$

From (4) and (6), the required general equation of the surfaces which are orthogonal to the given family of surfaces (1) is given by

$$(x^2 + y^2 + z^2)/z = \phi[(2x^2 + y^2)/z^2]$$

where ϕ is an arbitrary function.

Example 9. Find the surface which is orthogonal to the one parameter system $z = cxy(x^2 + y^2)$ which passes through the hyperbola $x^2 - y^2 = a^2 z = 0$.

Solution. The equation of given system of surfaces is

$$f(x, y, z) = z/(x^3 y + xy^3) = c \qquad \ldots(1)$$

$$\therefore \qquad \frac{\partial f}{\partial x} = -\frac{z(3x^2 y + y^3)}{(x^3 y + xy^3)^2}, \frac{\partial f}{\partial y} = -\frac{z(3y^2 x + x^3)}{(x^3 y + xy^3)^2}, \frac{\partial f}{\partial z} = \frac{1}{x^3 y + xy^3}$$

The required orthogonal surface will be the solution of

$$p\left(\frac{\partial f}{\partial x}\right) + q\left(\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial z} \quad \text{or} \quad -\frac{z(3x^2 y + y^3)}{(x^3 y + xy^3)^2} p - \frac{z(3y^2 x + x^3)}{(x^3 y + xy^3)^2} q = \frac{1}{x^3 y + xy^3}$$

or $\quad \{(3x^2 + y^2)/x\}p + \{(3y^2 + x^2)/y\}q = -(x^2 + y^2)/z \ \ldots(2)$

The Lagrange's auxiliary equations for (2) are

$$\frac{dx}{(3x^2 + y^2)/x} = \frac{dy}{(3y^2 + x^2)/y} = \frac{dz}{-(x^2 + y^2)/z} \qquad \ldots(3)$$

Taking the first two fractions of (3), we have

$$2xdx - 2ydy = 0 \quad \text{so that} \quad x^2 - y^2 = c_1 \qquad \ldots(4)$$

Choosing $x, y, 4z$ as multipliers, each fraction of (3) is

$$= (xdx + ydy + 4zdz)/0$$

$\therefore\ 2xdx + 2ydy + 8zdz = 0 \quad$ so that $\quad x^2 + y^2 + 4z^2 = c_2 \ \ldots(5)$

Hence, the surface which is orthogonal to (1) is given by

$$x^2 + y^2 + 4z^2 = \phi(x^2 - y^2) \qquad \ldots(6)$$

For the particular surface passing through the given hyperbola $x^2 - y^2 = a^2, z = 0$, we must take

$$\phi(x^2 - y^2) = a^2(x^2 + y^2)/(x^2 - y^2)^2 \qquad \ldots(7)$$

Hence, the required surface is given by

$$(x^2 + y^2 + 4z^2)^2(x^2 - y^2)^2 = a^4(x^2 + y^2) \qquad \ldots(8)$$

Example 10. Find the family orthogonal to $\phi[z(x+y)^2, x^2 - y^2] = 0$.

Solution. Given that $\qquad \phi[z(x+y)^2, x^2 - y^2] = 0 \qquad \ldots(1)$

Let $\quad u = z(x+y)^2 \quad$ and $\quad v = x^2 - y^2 \qquad \ldots(2)$

Then (1) becomes $\qquad \phi(u, v) = 0 \qquad \ldots(3)$

Differentiating (3) w.r.t. *x* and *y* partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots(4)$$

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad \ldots(5)$$

From (2), we get $\frac{\partial u}{\partial x} = 2z(x+y), \frac{\partial u}{\partial y} = 2z(x+y),$

$$\frac{\partial u}{\partial z} = (x+y)^2, \quad \frac{\partial v}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -2y, \quad \frac{\partial v}{\partial z} = 0$$

Putting these values in (4) and (5), we get

$$\left(\frac{\partial \phi}{\partial u}\right)[2z(x+y) + p(x+y)^2] + \left(\frac{\partial \phi}{\partial v}\right)(2x + 0) = 0 \qquad \ldots(6)$$

and $$\left(\frac{\partial \phi}{\partial u}\right)[2z(x+y) + q(x+y)^2] + \left(\frac{\partial \phi}{\partial v}\right)(-2y + 0) = 0 \qquad \ldots(7)$$

Evaluating the values of $\frac{\partial \phi}{\partial u}/\frac{\partial \phi}{\partial v}$ from (6) and (7) and then equating these, we get

$$\frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = \frac{2x}{2z(x+y)+p(x+y)^2} = \frac{-2y}{2z(x+y)+q(x+y)^2}$$

or $\quad x(x+y)[2z+q(x+y)] = -y(x+y)[2z+p(x+y)]$

or $\quad 2xz + qx(x+y) + 2yz + py(x+y) = 0$

or $\quad py(x+y) + qx(x+y) = -2z(x+y)$

or $\quad py + qx = -2z \qquad \ldots(8)$

which is a partial differential equation of the family of surfaces given by (1).

The differential equation of the family of surfaces orthogonal to (8) is given by

$$ydx + xdy - 2zdz = 0 \quad \text{or} \quad d(xy) - 2zdz = 0 \qquad \ldots(9)$$

Integrating (9), we get $\quad xy - z^2 = c \qquad \ldots(10)$

which is the desired family of orthogonal surfaces.

EXERCISE 2(E)

1. Find particular integrals of the following partial differential equations to represent surfaces passing through the given curves:

(i) $p + q = 1; x = 0, y^2 = z$

(ii) $xp + yq = z; x + y = 1, yz = 1$

(iii) $(y-z)p + (z-x)q = x - y; z = 0, z = 2x$

(iv) $x(y-z)p + v(z-x)q = z(x-y); x = y, x = y - z$

(v) $yp - 2xyq = 2xa; x = t, y = t^2, z = t^2$

(vi) $(y-z)[2xyp + (x^2-y^2)q] + z(x^2-y^2) = 0; x = t^2, y = 0, z = t^3$

2. Find the general solution of the partial differential equation $2x(y+z^2)p+(2y+z^2)q=z^2$ and deduce that $yz(z^2+yz-2y)=x^2$ is a solution.

3. Find the general solution of the partial differential equation $x(z+2ap)+(xz+2yz+2ay)q=z(z+a)$. Find also the integral surfaces which pass through the curves:

(i) $y=0, z^2=4ax$ **(ii)** $y=0, z^3+x(z+a)^2=0$

4. Solve $xp+yq=z$. Find a solution representing a surface meeting the parabola $y^2=4x, z=1$.

ANSWERS

1(i) $(y-x)^2=z-x$ **(ii)** $yz=(x+y)^2$

(iii) $5(x+y+z)^2=9(x^2+y^2+z^2)$ **(iv)** $(x+y+z)^3=27xyz$

(v) $(x^2+y^2)^3=32y^2z^2$ **(vi)** $x^3-3xy^2=z^2-2yz$

4. General Solution: $\phi\left(\frac{x}{2},\frac{y}{2}\right)=0$, Required Surface: $y^2=4xz$

2.17 Linear Partial Differential Equations of Order One with n Independent Variables

Let $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ be the n independent variables and z be a dependent function depending on $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$.

Also, let $p_1=\frac{\partial z}{\partial x_1}, p_2=\frac{\partial z}{\partial x_2}, p_3=\frac{\partial z}{\partial x_3}, \ldots, p_n=\frac{\partial z}{\partial x_n}$

Then, the general linear partial differential equation of order one with n independent variables is given by

$$P_1p_1+P_2p_2+P_3p_3+\ldots+P_np_n=R \qquad \ldots(1)$$

where $P_1, P_2, P_3, \ldots\ldots, P_n$ are the functions of $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ and R is a function of $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ and z.

The above partial differential equation (1) can be solved by the generalization of Lagrange's method. Therefore, the system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \ldots\ldots\ldots = \frac{dx_n}{P_n} = \frac{dz}{R} \qquad \ldots(2)$$

Let $u_1(x_1, x_2, x_3, \ldots, x_n, z) = c_1$, $u_2(x_1, x_2, x_3, \ldots, x_n, z) = c_2$, $u_3(x_1, x_2, x_3, \ldots, x_n, z) = c_3, \ldots$, $u_n(x_1, x_2, x_3, \ldots, x_n, z) = c_n$ be any n independent integrals of (2).

Then, the general solution of (1) is given by

$$\phi(u_1, u_2, u_3, \ldots \ldots, u_n) = 0 \qquad \ldots(3)$$

SOLVED EXAMPLES

Example 1. Solve $x_2x_3p_1 + x_3x_1p_2 + x_1x_2p_3 = x_1x_2x_3$.

Solution. The given equation is a linear partial differential equation with three independent variables x_1, x_2 and x_3 and z as a dependent function depending on x_1, x_2 and x_3.

Comparing the given partial differential equation with $P_1p_1 + P_2p_2 + P_3p_3 + \ldots + P_np_n = R$, we have

$$P_1 = x_2x_3, \quad P_2 = x_3x_1, \quad P_3 = x_1x_2 \text{ and } R = x_1x_2x_3$$

∴ The system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{p_1} = \frac{dx_2}{p_2} = \frac{dx_3}{p_3} = \frac{dz}{R} \text{ or } \frac{dx_1}{x_2x_3} = \frac{dx_2}{x_3x_1} = \frac{dx_3}{x_1x_2} = \frac{dz}{x_1x_2x_3} \qquad \ldots(1)$$

Taking the first and the second fractions of (1), we get

$$x_1dx_1 = x_2dx_2 \text{ so that } \frac{x_1^2}{2} = \frac{x_2^2}{2} + \frac{C_1}{2}$$

which gives $x_1^2 - x_2^2 = c_1$ or $u_1 \equiv x_1^2 - x_2^2 = c_1$ …(2)

Taking the second and the third fractions of (1), we get

$$x_2 dx_2 = x_3 dx_3 \quad \text{so that} \quad \frac{x_2^2}{2} = \frac{x_3^2}{2} + \frac{c_2}{2}$$

which give $x_2^2 - x_3^2 = c_2$ or $u_2 \equiv x_2^2 - x_3^2 = c_2$ …(3)

Again, taking the third and fourth fractions of (1), we get

$$dz = x_3 dx_3 \quad \text{so that} \quad z = \frac{x_3^2}{2} + \frac{c_3}{2}$$

which gives $2z - x_3^2 = c_3$ or $u_3 \equiv 2z - x_3^2 = c_3$ …(4)

Finally, from (2), (3) and (4), the general solution of the given partial differential equation is

$$\phi(x_1^2 - x_2^2, x_2^2 - x_3^2, 2z - x_3^2) = 0 \qquad \ldots(5)$$

Example 2. Solve $P_1 p_1 + P_2 p_2 + P_3 p_3 = az + \frac{x_1 x_2}{x_3}$.

Solution: The given equation is a linear partial differential equation with three independent variables x_1, x_2, x_3 and z as a dependent function depending on x_1, x_2 and x_3.

Comparing the given partial differential equation with $P_1 p_1 + P_2 p_2 + P_3 p_3 + \ldots = R$, we have

$$P_1 = x_1, P_2 = x_2, \ P_3 = x_3 \text{ and } R = az + \frac{x_1 x_2}{x_3}.$$

∴ The system of Lagrange's auxiliary equations is given by

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \frac{dz}{R} \quad \text{or} \quad \frac{dx_1}{x_1} = \frac{dx_2}{x_2} = \frac{dx_3}{x_3} = \frac{dz}{az + \frac{x_1 x_2}{x_3}} \qquad \ldots(1)$$

Taking the first and the second fractions of (1), we have

$$\frac{dx_1}{x_1} = \frac{dx_2}{x_2} \quad \text{so that} \quad \log x_1 = \log x_2 + \log c_1$$

$$\therefore \qquad \frac{x_1}{x_2} = c_1 \qquad \text{i.e.} \qquad u_1 = \frac{x_1}{x_2} = c_1 \qquad \ldots(2)$$

Taking the second and the third fractions of (1), we have

$$\frac{dx_2}{x_2} = \frac{dx_3}{x_3} \quad \text{so that} \quad \log x_2 = \log x_3 + \log c_2$$

$$\therefore \qquad \frac{x_2}{x_3} = c_2 \qquad \text{i.e.} \qquad u_2 = \frac{x_2}{x_3} = c_2 \qquad \ldots(3)$$

Again, taking the first and fourth fractions of (1), we have

$$\frac{dx_1}{x_1} = \frac{dz}{az + \frac{x_1 x_2}{x_3}} = \frac{dz}{az + c_2 x_1}, \text{ since } \frac{x_2}{x_3} = c_2$$

or $$\frac{az + c_2 x_1}{x_1} = \frac{dz}{dx_1} \quad \text{i.e.,} \quad \frac{dz}{dx_1} - \left(\frac{a}{x_1}\right) z = c_2 \qquad \ldots(4)$$

which is a linear differential equation whose integrating function (I.F.) is given as follows :

$$\text{I.F. of (4)} = e^{-a \int \frac{dx_1}{x_1}} = e^{-a \log x_1} = x_1^{-a}$$

∴ The solution of the linear differential equation (4) is given by

$$z x_1^{-a} = c_2 \int x_1^{-a} \, dx_1 + c_3 \quad \text{or} \quad z x_1^{-a} = c_2 \left(\frac{x_1^{1-a}}{1-a}\right) + c_3$$

or $$z x_1^{-a} = \frac{x_2}{x_3} . \frac{x_1^{1-a}}{(1-a)} + c_3, \text{ since from (2), } c_2 = \frac{x_2}{x_3}$$

$$\therefore \quad \frac{z}{x_1^a} - \left(\frac{x_1^{1-a}}{1-a}\right)\frac{x_2}{x_3} = c_3 \ \text{ i.e. } \ u_3 = \frac{z}{x_1^a} - \left(\frac{x_1^{1-a}}{1-a}\right)\frac{x_2}{x_3} = c_3 \quad \ldots(5)$$

Finally, from (2), (3) and (5), the general solution of the given partial differential equation is

$$\phi\left[\frac{x_1}{x_2},\frac{x_2}{x_3},\left\{\frac{z}{x_1^a}-\left(\frac{x_1^{1-a}}{1-a}\right)\frac{x_2}{x_3}\right\}\right]=0 \qquad \ldots(6)$$

EXERCISE 2(F)

Solve the following partial differential equations:

1. $x_1p_1+x_2p_2+x_3p_3=x_1x_2x_3$.
2. $(x_3-x_2)p_1+x_2p_2-x_3p_3=x_2(x_1+x_3)-x_2^2$.
3. $p_1-x_1p_2+x_1x_2p_3+x_1x_2x_3\sqrt{z}=0$.
4. $(x_2+x_3+z)p_1+(x_3+x_1+z)p_2+(x_1+x_2+z)p_3$ $=x_1+x_2+x_3$.

ANSWERS

1. $\phi\left(\frac{x_2}{x_1},\frac{x_1}{x_3},x_1x_2x_3-3z\right)=0$
2. $\phi(z-x_1x_2,x_1+x_2+x_3,x_2x_3)=0$
3. $\phi\left(2x_2+x_1^2,2x_3+x_2^2,4\sqrt{z}+x_3^2\right)=0$
4. $\phi\{u(z-x_1),u(z-x_2),u(z-x_3)\}=0$, where u is given by $u=(z+x_1+x_2+x_3)^{1/3}$

OBJECTIVE TYPE QUESTIONS

1. The PDE $Pp+Qq=R$ is popularly known as

(a) Lagrange's equation (b) Euler's equation

(c) Monge's equation (d) Leibnitz equation

2. Lagrange's auxiliary equations for $xzp+yzq=xy$ are

(a) $\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{xy}$ (b) $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$

(c) $\frac{dx}{p}=\frac{dy}{q}=\frac{dz}{1}$ (d) $\frac{dx}{p}=\frac{dy}{q}=\frac{dz}{z}$

3. The integral surface satisfying $4yzp + q + 2y = 0$ and passing through $y^2 + z^2 = 1, x + z = 2$ is

(a) $y^2 + z^2 + x + z - 3 = 0$ (b) $y^2 + z^2 + x + z = 0$

(c) $y^2 + z^2 + y + z - 3 = 0$ (d) $y^2 + z^2 + y + z = 0$

4. The solution of the PDE $xzp + yzq = xy$ is

(a) $\phi\left(\frac{x}{y}, xy - z^2\right) = 0$ (b) $\phi(x^2, xy) = 0$

(c) $\phi(x^2y, xy) = 0$ (d) $\phi(xy, z + x^2y) = 0$

ANSWERS

1. (a) **2.** (a) **3.** (a) **4.** (a)