Math 2141: Practice Problems on Mean Value Theorem for Exam 2

These problems are to give you some practice on using Rolle's Theorem and the Mean Value Theorem for Exam 2. You do not need to hand them in.

Problem 1. For each of the following functions, verify that they satisfy the hypotheses of Rolle's Theorem on the given intervals and find all points c in the given interval for which $f'(c) = 0$.

$$f(x) = 2x^2 - 4x + 5 \text{ on } [-1, 3]$$
$$g(x) = x^3 - 2x^2 - 4x + 2 \text{ on } [-2, 2]$$

Problem 2. Let $f(x)$ be a function which is continuous on $[a, b]$, differentiable on (a, b) and such that $f(a)$ and $f(b)$ have different signs (i.e. one is strictly positive and the other is strictly negative). Prove that if $f'(x)$ is either strictly positive or strictly negative on (a, b), then the equation $f(x) = 0$ has exactly one solution in the interval $[a, b]$.

Hint. First, you need to explain why $f(x) = 0$ has at least one solution in $[a, b]$. (This is an immediate consequence of a theorem you know.) Second, you need to prove that $f(x) = 0$ cannot have two solutions in $[a, b]$. I would do this part by contradiction. That is, assume that there are $c_1 \neq c_2$ in $[a, b]$ such that $f(c_1) = f(c_2) = 0$. Use Rolle's Theorem to get a contradiction.

Problem 3. Let $f(x) = x^3 - 3x + 1$. Use Problem 2 to explain why there is exactly one point $c \in [-1, 1]$ such that $f(c) = 0$.

Problem 4. Check that $f(x) = x^2 + 4x - 1$ satisfies the conditions of the Mean Value Theorem on the interval $[0, 2]$ and find all values c such that $f'(c)$ is equal to the slope of the secant line connecting the points $(0, f(0))$ and $(2, f(2))$.

In Problem 4, you should have found a single value for c and it should be the midpoint of the interval $[0, 2]$. In the next problem, you will show that as long as $f(x)$ is a quadratic function, the value c in the Mean Value Theorem applied to $f(x)$ on $[a, b]$ is always the midpoint of $[a, b]$. This property is a special property of quadratic functions!

Problem 5. Let $f(x) = \alpha x^2 + \beta x + \gamma$ be a quadratic function where $\alpha, \beta, \gamma \in \mathbb{R}$. Consider the closed interval $[a, b]$ with midpoint $c = (a + b)/2$. Prove that the slope of the secant line joining the points $(a, f(a))$ and $(b, f(b))$ is equal to the slope of the tangent line to $f(x)$ at c.

Note. This question is really asking you to show that the midpoint c is the same as the point c specified by the Mean Value Theorem for Derivatives applied to $f(x)$ on the interval $[a, b]$.

Problem 6. Let $f(x) = x^{2/3}$ and note that $f(-1) = f(1) = 1$. Show that there is no $c \in [-1, 1]$ for which $f'(c) = 0$. Why doesn't this contradict Rolle's Theorem?

Problem 7. Let $0 < a < b$ be real numbers and let $n \geq 2$ be an integer. Use the Mean Value Theorem to explain why

$$na^{n-1}(b-a) \leq b^n - a^n \leq nb^{n-1}(b-a)$$

Hint. Let c be the point in the conclusion of the Mean Value Theorem applied to the function $f(x) = x^n$ on the interval $[a, b]$. Using the fact that $f(x)$ is an increasing function, what is biggest $f'(c)$ could be? What is the smallest $f'(c)$ could be?

Problem 8. Let $f(x)$ be a function which is continuous on $[a, b]$ and such that $f''(x)$ exists at every point in (a, b). Suppose that the line segment joining $(a, f(a))$ and $(b, f(b))$ intersects the graph of $f(x)$ at a point $(c, f(c))$ where $a < c < b$. Prove that there is at least one point $d \in (a, b)$ at which $f''(d) = 0$.

Hint. Use the Mean Value Theorem to show that there are distinct points $c_0, c_1 \in (a, b)$ such that $f'(c_0) = f'(c_1)$. Now use Rolle's Theorem to get a point d such that $f''(d) = 0$.

We proved two parts of the last problem in class. As this is one of the most important applications of the Mean Value Theorem in calculus, it is well worth reviewing this material.

Problem 9. Let $I = (a, b)$ be an open interval and let f be a function which is differentiable on I. Use the Mean Value Theorem to prove the following statements.

9(a). If $f'(x) = 0$ for all $x \in I$, then there is a constant r such that $f(x) = r$ for all $x \in I$.

9(b). If $f'(x) > 0$ for all $x \in I$, then $f(x)$ is strictly increasing on I.

9(c). If $f'(x) < 0$ for all $x \in I$, then $f(x)$ is strictly decreasing on I.